COMPREHENSIVE REVIEW OF DOUBLE INTEGRAL TRANSFORMS.

Abstract:

Sumudu-Shehu transform, Sawi-ARA transform, In this paper we have presented a meticulous comparative review of double integral transforms: namely Fourier-Laplace transform, Laplace-ARA transform, Laplace-Sawi transform, Sumudu-Sawi transform, Sumudu-ARA transform, Sumudu-Shehu transform, Sawi-ARA transform. Here integral transform of basic functions is compared along their differential properties. Further more we also highlighted special advantages of these double integral transforms and their applications in different areas of sciences and technology.

Key words:-

Fourier-Laplace transform, Laplace-ARA transform, Laplace-Sawi transform, Sumudu- Sawi transform, Sumudu -ARA Transform,

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Introduction:-

Integral transforms provide us a unique mathematical technique which helps us to convert analytical complex problems into simple algebraic ones. Based on unique features of each integral transform, their double integral transformed proposed with their wide applications in different areas of the sciences and technology. One-dimensional wave equation, one-dimensional heat flow equation is solved by using these double Fourier-Laplace Transform by V. D. Sharma and A.N. Rangari [6]. Sedeeg, A.K.; Mahamoud, Z.I.; Saadeh, R. presented solution of homogeneous Laplace equation, homogeneous Telegraph equation, homogeneous heat equation, Klein-Gordon equation, advection-diffusion equation.

The Goursat equation with the help of double Laplace -ARA transform [7]. Raed R. Abu Awwad, Monther AI-Momani, Baha Abughaleh, Ali Jaradat, Abdul Karim Farah presented application of Sumudu-Sawi transform for finding solution of Klein -Gordon equation, telegraph equation, Voltera-integro parial differential equation [8]. Double ARA-Sumudu transform used to solve wave equation, telegraph equation, Klein-Gordon equation by Rania Saadeh, Ahmad Qaa and Aliaa Burqan [9]. ARA-Sawi Transform its application to partial differential equation demonstrated by Raed R. Abu Awwad, Monther Al-Momani, Ali -Jaradat, Baha Abughaaleh, Ahmad AI-Natoor [10]. Application of Combined Sumudu-Shehu transform discussed by Monther AI-Momani, Ali Jaradat, Baha Abughazaleh, Abdul Karim Farah [11]. Monther AI-Momani, Ali Jaradat, Baha Abughazaleh also presented application of double Laplace-Sawi transform to solve partial differential equation.

This paper is having following sections; in section 2 we have given definitions of integral transforms which show analysis of kernel. Section 3 presents Definition and properties of double Integral transforms and also their derivative formulas, Section 4 contain key theoretical and behavioral characteristics of integral transform. Section 5 highlights Special Advantages and Section 6 present Applications. This paper provides comprehensive, critical and comparative analysis.

2. Preliminaries and Kernel Analysis:

Let A be the set of functions $f(t)$ defined for $t \geq 0$ which are of exponential order, meaning $|f(t)| \leq M e^{kt}$ for some constant M, k.

Now here we define some double integral transforms under review.

2.1 Laplace Transform:

Laplace transform of function $f(t)$ is defined as

$$L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)\, \mathrm{d}t \quad , \ Re(s) > 0$$

2.2 Fourier Transform:

The Fourier transform was invented by French mathematician and Physicist Jean-Baptiste Joseph Fourier in the early 19^{th} century. He first presented his groundbreaking ideas in 1807 and later published them in his 1822 workbook.

Fourier transform of function $f(t)$ is defined as

$$F\{f(t)\} = \int_{-\infty}^{\infty} e^{-iwt} f(t) \quad \mathrm{d}t$$

2.3 The Sumudu Transform :

In the early 1990s, new integral transform is introduced by Sri Lankan Engineer and mathematician G.K. Watugala while working at the department of Electrical and Electronics Engineering at the University of Peradeniya, Sri Lanka , which preserve Unit property.

Sumudu transform of function $f(t)$ is defined as,

$$S\{f(t)\} = \frac{1}{k} \int_{0}^{\infty} e^{-\frac{t}{k}} f(t) \ \mathrm{d}t$$

Where, k has unit of time.

2.4 Shehu Transform:

Shehu Maitama and Weidong zhao in 2019 introduced Shehu transform which generalizes both the Laplace transform, which is applied to both ordinary and partial differential equations.

Shehu transform of function $f(t)$ is defined as,

$$\mathbb{S}\{f(t)\} = \int_{0}^{\infty} e^{\frac{-st}{u}} f(t) \ \mathrm{d}t$$

2.5 ARA Transform:

The ARA integral Transform was introduced in 2020 by mathematician R. Saadeh and Ahmad Qazza. This transform is used to simplify and solve complex differential and integral equations used in engineering and physics.

The ARA transform of continuous function $f(t)$ defined as,

$$\mathcal{G}\{f(t)\}(s) = s \int_{0}^{\infty} e^{-st} f(t) \ \mathrm{d}t \quad , \quad s > 0$$

2.6 Sawi Transform:

A new transform called Sawi transform was introduced in 2019 by mathematician Mohand Mahgoub. The transform is defined for $f(t)$ as follows,

$$W\{f(t)\}=\frac{1}{\nu^2}\int_0^\infty f(t)\ \mathrm{d}t$$

3. Double integral Transform:

In this section we discuss double integral transform of all above-mentioned integral transforms and their basic properties and differential properties.

3.1 Fourier-Laplace Transform:

Fourier-Laplace Transform for the function $f(t,x)$ is defined as

$$FL\{f(t,x)\}=F(s,p)=\int_{-\infty}^{\infty}\int_0^\infty e^{-i(st-ipx)}\ f(t,x)\ \mathrm{d}t\mathrm{d}x$$

3.1.1 Fourier-Laplace transforms of some basic functions:

Sr. No.	$f(t,x)$	$FL\{f(t,x)\}=F(s,p)$
1	**1**	$\frac{-i}{sp}$
2	$t\ x$	$\frac{-1}{s^2p^2}$
3	$t^n x^n$	$\frac{(-1)^{-n-1}(i)^{-n-1}(n!)^2}{s^{n+1}p^{n+1}}$
4	e^{at+bx}	$\frac{1}{(is-a)(p-b)}$
5	$Sinat\ \sin bx$	$\frac{ab}{(a^2-s^2)(b^2-p^2)}$
6	$Cosat\ \cos bx$	$\frac{isp}{(a^2-s^2)(b^2-p^2)}$
7	$Sinhat\ \sinh bx$	$\frac{-ab}{(s^2+a^2)(p^2-b^2)}$
8	$Coshat\ \cosh bx$	$\frac{-isp}{(s^2+a^2)(p^2-b^2)}$

3.1.2 Differential Property:

$$1.\ FLf_{x}(t,x)=pFLf(t,x)-k \quad 2.\ FLf_{xx}(t,x)=p^{2}FLf(t,x)-pk$$

$$3.\ FLf^{n}_{x}(t,x)=p^{n}FLf(t,x)-p^{n-1}k$$

Where $k=\int_{-\infty}^{\infty} e^{-ist f(t,0)}\,\mathrm{d}t$

3.1.3 First Shifting theorem:

$$FL\{e^{-ax-bt}f(t,x)\}=F(s-ib\,,\,p+a)$$

3.2 Laplace -ARA Transform:

Let $u(x,t)$ have a continuous function of two positive variables x & t. Then Laplace-ARA transform of $u(x,t)$ is defined as

$$L_{x}\mathcal{G}_{t}\{u(x,t)\}=Q(\nu,s)=s\int_{0}^{\infty}\int_{0}^{\infty} e^{-(\nu x+st)}\,u(x,t)\,\mathrm{d}x\mathrm{d}t\ , \qquad \nu,s>0$$

Provided integral exists.

3.2.1 Laplace -ARA Transform of some basic functions:

Sr. No.	$u(x,t)$	$L_{x}\mathcal{G}_{t}\{u(x,t)\}=Q(\nu,s)$
1	**1**	$\frac{1}{\nu}$
2	$x^{\alpha}t^{\beta}$	$\frac{\alpha!\ \beta!}{\nu^{\alpha+1}s^{\beta}}\ ,\quad \alpha,\beta>-1$
4	$(e^{ax+bt}$	$\frac{s}{(\nu-\alpha)(s-\beta)}$
5	$Sin(\alpha x+\beta t)$	$\frac{s(\nu\beta+s\alpha)}{(\nu^{2}+\alpha^{2})(s^{2}+\beta^{2})}$
6	$Cos(\alpha x+\beta t)$	$\frac{s(s\nu-\alpha\beta)}{(\nu^{2}+\alpha^{2})(s^{2}+\beta^{2})}$
7	$Sinh(\alpha x+\beta t)$	$\frac{s(\nu\beta+s\alpha)}{(\nu^{2}-\alpha^{2})(s^{2}-\beta^{2})}$
8	$Cosh(\alpha x+\beta t)$	$\frac{s(s\nu+\alpha\beta)}{(\nu^{2}-\alpha^{2})(s^{2}-\beta^{2})}$

3.2.2 Differential Property:

$$1.\ L_x\,\mathscr{G}_t\left\{u_t(x,t)\right\}=s\,Q(\nu,s)-s\,L\{u(x,0)\}$$

$$2.\ L_x\,\mathscr{G}_t\{u_x(x,t)\}=\nu\,Q(\nu,s)-\mathscr{G}\{u(0,t)\}$$

$$3.\ L_x\,\mathscr{G}_t\left\{u_{tt}(x,t)\right\}=s^2\,Q(\nu,s)-s^2\,L\{u(x,0)\}-s\,L\left\{u_t(x,0)\right\}$$

$$4.\ L_x\,\mathscr{G}_t\left\{u_{xx}(x,t)\right\}=\nu^2\,Q(\nu,s)-\nu\,\mathscr{G}\{u(0,t)\}-\mathscr{G}\left\{u_x(0,t)\right\}$$

$$5.\ L_x\,\mathscr{G}_t\{u_{xt}(x,t)\}=\nu s\,Q(\nu,s)-s\,\mathscr{G}\{u(0,t)\}-\nu s\,L\{u(x,0)\}+s\,u(0,0)$$

3.2.3 First Shifting theorem:

$$L_x\,\mathscr{G}_t\{e^{\alpha x+\beta t}\,u(x,t)\}=\frac{s}{(s-\beta)}Q(\nu-\alpha,\ s-\beta)$$

3.3 Sumudu- Sawi Transform:

The Sumudu - Sawi transform of function $h(\xi,x)$ defined as,

$$S_\xi\,W_x\{h(\xi,x)\}=H(\delta,\varepsilon)=\frac{1}{\delta\,\varepsilon^2}\int_0^\infty\int_0^\infty e^{-\left(\frac{\xi}{\delta}-\frac{x}{\varepsilon}\right)}h(\xi,x)\,\mathrm{d}\xi\,\mathrm{d}x\ ,$$

Provided integral exists.

3.3.1 Sumudu-Sawi Transform of some basic functions:

Sr. No.	$h(\xi,x)$	$S_\xi W_x\{h(\xi,x)\}=H(\delta,\varepsilon)$
1	**1**	$\frac{1}{\varepsilon}$
2	$\xi^\mu\,x^\nu$	$\delta^\mu\,\varepsilon^{\nu-1}\ \ \mu!\ \ \nu!\qquad Re\left(\frac{1}{\delta}\right)>0\,,\quad Re(\mu)>-1$
4	$e^{\mu\xi+\nu x}$	$\frac{1}{\varepsilon\,(1-\delta\mu)\,(1-\nu\varepsilon)},\quad Re\left(\frac{1}{\delta}\right)>Re(\mu)$
5	$e^{i(\mu\xi+\nu x)}$	$\frac{-1}{\varepsilon\,(i+\delta\mu)\,(i+\nu\varepsilon)},\quad Re\left(\frac{1}{\delta}\right)+Img(\mu)>0$

6	$Sin(\mu\xi+\nu x)$	$\dfrac{(\mu\delta+\varepsilon\nu)}{\varepsilon\left(1+\mu^2\delta^2\right)\left(1+\nu^2\varepsilon^2\right)}$ $\lvert Img(\mu)\rvert < Re\left(\dfrac{1}{\delta}\right)$
6	$Cos(\mu\xi+\nu x)$	$\dfrac{(1-\mu\delta\varepsilon\nu)}{\varepsilon\left(1+\mu^2\delta^2\right)\left(1+\nu^2\varepsilon^2\right)}$ $\lvert Img(\mu)\rvert < Re\left(\dfrac{1}{\delta}\right)$
7	$Sinh(\mu\xi+\nu x)$	$\dfrac{(\mu\delta+\varepsilon\nu)}{\varepsilon\left(1-\mu^2\delta^2\right)\left(1-\nu^2\varepsilon^2\right)}$ $Re\left(\dfrac{1}{\delta}\right) > Re(\mu) \quad \& \quad Re\left(\dfrac{1}{\delta}+\mu\right) > 0$
8	$Cosh(\mu\xi+\nu x)$	$\dfrac{(1+\mu\delta\varepsilon\nu)}{\varepsilon\left(1-\mu^2\delta^2\right)\left(1-\nu^2\varepsilon^2\right)}$ $Re(\mu) < Re\left(\dfrac{1}{\delta}\right) \quad \& \quad Re\left(\dfrac{1}{\delta}+\mu\right) > 0$

3.3.2 Differential Property:

$$1.\ S_{\xi}W_{x}\left\{h_{\xi}(\xi,x)\right\}=\frac{1}{\delta}H(\delta,\varepsilon)-\frac{1}{\delta}W\{h(0,x)\}$$

$$2.\ S_{\xi}W_{x}\left\{h_{\xi\xi}(\xi,x)\right\}=\frac{1}{\delta^2}H(\delta,\varepsilon)-\frac{1}{\delta^2}W\{h(0,x)\}-\frac{1}{\delta}W\left\{h_{\xi}(0,x)\right\}$$

$$3.\ S_{\xi}W_{x}\{h_{x}(\xi,x)\}=\frac{1}{\varepsilon}H(\delta,\varepsilon)-\frac{1}{\varepsilon^2}S\{h(\xi,0)\}$$

$$4.\ S_{\xi}W_{x}\{h_{xx}(\xi,x)\}=\frac{1}{\varepsilon^2}H(\delta,\varepsilon)-\frac{1}{\varepsilon^2}S\{h(\xi,0)\}-\frac{1}{\varepsilon^2}S\{h_{x}(\xi,0)\}$$

$$5.\ S_{\xi}W_{x}\{h_{\xi x}(\xi,x)\}=\frac{1}{\varepsilon\delta}H(\delta,\varepsilon)-\frac{1}{\delta\varepsilon^2}S\{h(\xi,0)\}-\frac{1}{\delta\varepsilon}W\{h(0,x)\}+\frac{1}{\delta\varepsilon^2}h(0,0)$$

3.4 Sumudu-Shehu Transform:

The Sumudu-Shehu transform of function $q(r,\nu)$ defined as,

$$S_{r}H_{\nu}\{q(r,\nu)\}=Q(k,\lambda,\mu)=\frac{1}{k}\int_0^{\infty}\int_0^{\infty}e^{-\left(\frac{r}{k}-\frac{\lambda}{\mu}\nu\right)}q(r,\nu)\,\mathrm{d}r\,\mathrm{d}\nu\ ,$$

Provided integral exists.

3.4.1 Sumudu-Shehu Transform of some basic functions:

Sr. No.	$q(r,\nu)$	$S_r H_\nu\{q(r,\nu)\} = Q(k,\lambda,\mu)$
1	**1**	$\frac{\mu}{\lambda}$
2	$r^\beta \nu^\tau$	$\frac{k^\beta \mu^{\tau+1} \beta! \tau!}{\lambda^{\tau+1}} \quad Re(k) > 0, \quad Re(\beta) > -1$
4	$e^{\beta r + \tau\nu}$	$\frac{\mu}{(1-k\beta)(\lambda-\tau\mu)}, \quad Re\left(\frac{1}{k}\right) > Re(\beta)$
5	$e^{i(\beta r + \tau\nu)}$	$\frac{i\mu}{(i-k\beta)(\lambda-i\tau\mu)}, \quad Re\left(\frac{1}{k}\right) + Img(\beta) > 0$
6	$Sin(\beta r + \tau\nu)$	$\frac{\mu(k\lambda\beta+\mu\tau)}{\varepsilon\left(1+k^2\beta^2\right)\left(\lambda^2+\tau^2\mu^2\right)}$ $\lvert Img(\beta)\rvert < Re\left(\frac{1}{k}\right)$
6	$Cos(\beta r + \tau\nu)$	$\frac{\mu(\lambda-k\mu\beta\tau)}{\left(1+k^2\beta^2\right)\left(\lambda^2+\tau^2\mu^2\right)}$ $\lvert Img(\beta)\rvert < Re\left(\frac{1}{k}\right)$
7	$Sinh(\beta r + \tau\nu)$	$\frac{\mu(k\lambda\beta+\mu\tau)}{\varepsilon\left(k^2\beta^2-1\right)\left(\lambda^2-\tau^2\mu^2\right)}$ $Re\left(\frac{1}{k}\right) > Re(\beta) \quad \& \quad Re\left(\frac{1}{k}+\beta\right) > 0$
8	$Cosh(\beta r + \tau\nu)$	$\frac{\mu(\lambda+k\mu\beta\tau)}{\left(k^2\beta^2-1\right)\left(\lambda^2-\tau^2\mu^2\right)}$ $Re\left(\frac{1}{k}\right) > Re(\beta) \quad \& \quad Re\left(\frac{1}{k}+\beta\right) > 0$

3.4.2 Differential Property:

1. $S_r H_\nu\left\{q_r(r,\nu)\right\} = \frac{1}{k} Q(k,\lambda,\mu) - \frac{1}{k} H\{q(0,v)\}$

2. $S_r H_\nu\left\{q_{rr}(r,\nu)\right\} = \frac{1}{k^2} Q(k,\lambda,\mu) - \frac{1}{k^2} H\{q(0,v)\} - \frac{1}{k} H\left\{q_r(0,v)\right\}$

3. $S_r H_\nu\{q_\nu(r,\nu)\} = \frac{\lambda}{\mu} Q(k,\lambda,\mu) - S\{q(r,0)\}$

4. $S_r H_\nu\left\{q_{\nu\nu}(r,\nu)\right\} = \frac{\lambda^2}{\mu^2} Q(k,\lambda,\mu) - \frac{\lambda}{\mu} S\{q(r,0)\} - S\left\{q_\nu(r,0)\right\}$

$$5.\ S_r H_\nu\left\{q_{r\nu}(r,\nu)\right\} = \frac{\lambda}{k\mu} Q(k,\lambda,\mu) - \frac{1}{k} S\{q(r,0)\} - \frac{\lambda}{k\mu} H\{q(0,\nu)\} + \frac{1}{k}\, q(0,0)$$

3.5 Laplace - Sawi Transform:

The Laplace - Sawi Transform of function $g(\eta,\theta)$ defined as,

$$L_\eta W_\theta\{g(\eta,\theta)\} = G(\delta,\varepsilon) = \frac{1}{\varepsilon^2}\int_0^\infty \int_0^\infty e^{-\left(\delta\eta - \frac{\theta}{\varepsilon}\right)} g(\eta,\theta)\, \mathrm{d}\eta\, \mathrm{d}\theta\ ,$$

Where $g(\eta,\theta)$ is continuous function on $(0,\infty) \times (0,\infty)$.

3.5.1 Laplace- Sawi Transform of some basic functions:

Sr. No.	$g(\eta,\theta)$	$L_\eta W_\theta\{g(\eta,\theta)\} = G(\delta,\varepsilon)$
1	**1**	$\frac{1}{\delta\varepsilon}$
2	$\eta^\mu\, \theta^\nu$	$\frac{\varepsilon^{\nu-1}\ \mu!\ \ \nu!}{\delta^{\mu+1}}$, $Re(\delta) > 0$, $Re(\mu) > -1$
4	$e^{\mu\eta+\nu\theta}$	$\frac{1}{\varepsilon\ (\delta-\mu)\ (1-\nu\varepsilon)}$, $Re(\delta) > Re(\mu)$
5	$e^{i(\mu\eta+\nu\theta)}$	$\frac{i}{\varepsilon\ (\delta-i\mu)\ (i+\nu\varepsilon)}$, $Re(\delta) + Img(\mu) > 0$
6	$Sin(\mu\eta+\nu\theta)$	$\frac{(\mu+\delta\varepsilon\nu)}{\varepsilon\left(\delta^2+\mu^2\right)\left(1+\nu^2\varepsilon^2\right)}$ $Img(\mu) < Re(\delta)$
6	$Cos(\mu\eta+\nu\theta)$	$\frac{(\delta+\varepsilon\mu\nu)}{\varepsilon\left(\delta^2+\mu^2\right)\left(1+\nu^2\varepsilon^2\right)}$ $Img(\mu) < Re(\delta)$
7	$Sinh(\mu\eta+\nu\theta)$	$\frac{(\mu+\delta\varepsilon\nu)}{\varepsilon\left(\delta^2-\mu^2\right)\left(1-\nu^2\varepsilon^2\right)}$ $Re(\mu) < Re(\delta)\ \ and\ \ Re(\mu) + Re(\delta) > 0$
8	$Cosh(\mu\eta+\nu\theta)$	$\frac{(\delta+\varepsilon\mu\nu)}{\varepsilon\left(\delta^2-\mu^2\right)\left(1-\nu^2\varepsilon^2\right)}$

3.5.2 Differential Property:

$$1.\ L_\eta W_\theta\left\{g_\eta(\eta,\theta)\right\} = \delta\, G(\delta,\varepsilon) - W\{g(0,\theta)\}$$

$$2.\ L_\eta W_\theta\left\{g_{\eta\eta}(\eta,\theta)\right\} = \delta^2 G(\delta,\varepsilon) - \delta\, W\{g(0,\theta)\} - W\left\{g_n(0,\theta)\right\}$$

$$3.\ L_\eta W_\theta\left\{g_\theta(\eta,\theta)\right\} = \frac{1}{\varepsilon} G(\delta,\varepsilon) - \frac{1}{\varepsilon^2} L\{g(\eta,0)\}$$

$$4.\ L_\eta W_\theta\left\{g_{\theta\theta}(\eta,\theta)\right\} = \frac{1}{\varepsilon^2} G(\delta,\varepsilon) - \frac{1}{\varepsilon^3} L\{g(\eta,0)\} - \frac{1}{\varepsilon^2} L\left\{g_\theta(\eta,0)\right\}$$

$$5.\ L_\eta W_\theta\{g_{\eta\theta}(\eta,\theta)\} = \frac{\delta}{\varepsilon} G(\delta,\varepsilon) - \frac{\delta}{\varepsilon^2} L\{g(\eta,0)\} - \frac{1}{\varepsilon} W\{g(0,\theta)\} + \frac{1}{\varepsilon^2} g(0,\theta)$$

3.6 ARA - Sawi Transform:

The ARA Sawi Transform of function $g(\rho,\sigma)$ defined as,

$$A_\rho W_\sigma\{g(\rho,\sigma)\} = G(\lambda,w) = \frac{\lambda}{w^2}\int_0^\infty\int_0^\infty e^{\left(-\lambda\rho-\frac{\sigma}{w}\right)} g(\rho,\sigma)\,d\rho\,d\sigma\ ,$$

Where $g(\rho,\sigma)$ is continuous function on $(0,\infty)\times(0,\infty)$.

3.6.1 ARA- Sawi Transform of some basic functions:

Sr. No.	$g(\rho,\sigma)$	$A_\rho W_\sigma\{g(\rho,\sigma)\} = G(\lambda,w)$
1	**1**	$\frac{1}{w}$, $Re(\lambda) > 0$
2	$\rho^r \sigma^\delta$	$\frac{w^{\delta-1}\ r!\ \delta!}{\lambda^r}$, $Re(\lambda) > 0$, $Re(r) > -1$
3	$e^{r\rho+\delta\sigma}$	$\frac{\lambda}{w(\lambda-r)(1-\delta w)}$, $Re(\lambda) > Re(r)$
4	$e^{i(r\rho+\delta\sigma)}$	$\frac{i\lambda}{w(\lambda-ir)(i-\delta w)}$, $Re(\lambda) + Img(r) > 0$
5	$Sin(r\rho+\delta\sigma)$	$\frac{\lambda(r+\lambda w\delta)}{w\left(\lambda^2+r^2\right)\left(1+\delta^2 w^2\right)}$ $Img(r) < Re(\lambda)$
6	$Cos(r\rho+\delta\sigma)$	$\frac{\lambda(\lambda-rw\delta)}{w\left(\lambda^2+r^2\right)\left(1+\delta^2 w^2\right)}$ $\lvert Img(r)\rvert < Re(\lambda)$
7	$Sinh(r\rho+\delta\sigma)$	$\frac{\lambda(r+\lambda w\delta)}{w\left(\lambda^2-r^2\right)\left(1-\delta^2 w^2\right)}$ $Re(r) < Re(\lambda)$ and $Re(r) + Re(\lambda) > 0$

8	$Cosh(r\rho+\delta\sigma)$	$\dfrac{\lambda(\lambda+r\,w\delta)}{w\left(\lambda^2-r^2\right)\left(1-\delta^2w^2\right)}$ $Re(r)<Re(\lambda)$ and $Re(r)+Re(\lambda)>0$

3.6.2 Differential Property:

$$1.\ A_{\rho}W_{\sigma}\left\{g_{\rho}(\rho,\sigma)\right\}=\lambda\,G(\lambda,w)\ -\lambda\,W\{g(0,\sigma)\}$$

$$2.\ A_{\rho}W_{\sigma}\{g_{\rho\rho}(\rho,\sigma)\}=\lambda^2\,G(\lambda,w)\ -\lambda^2\,W\{g(0,\sigma)\}-\lambda\,W\{g_{\rho}(0,\sigma)\}$$

$$3.\ A_{\rho}W_{\sigma}\left\{g_{\sigma}(\rho,\sigma)\right\}=\frac{1}{w}\,G(\lambda,w)\ -\frac{1}{w^2}\,A\{g(\rho,0)\}$$

$$4.\ A_{\rho}W_{\sigma}\{g_{\sigma\sigma}(\rho,\sigma)\}=\frac{1}{w^2}\,G(\lambda,w)\ -\frac{1}{w^3}\,A\{g(\rho,0)\}-\frac{1}{w^3}\,A\{g_{\sigma}(\rho,0)\}$$

$$5.\ A_{\rho}W_{\sigma}\{g_{\rho\sigma}(\rho,\sigma)\}=\frac{\lambda}{w}\,G(\lambda,w)\ -\frac{\lambda}{w^2}\,A\{g(\rho,0)\}-\frac{\lambda}{w}\,W\{g(0,\sigma)\}+\frac{\lambda}{w^2}g(0,0)$$

3.7 ARA- Sumudu Transform:

The ARA-Sumudu Transform of function $g(x,y)$ defined as,

$$\mathscr{G}x\,S_{y}\{g(x,y)\}=G(s,\mu)\ =\ \frac{1}{\mu}\int_0^{\infty}\int_0^{\infty}e^{\left(-sx-\frac{y}{\mu}\right)}\,g(x,y)\,\mathrm{d}x\ \mathrm{d}y\ ,\qquad s>0\ ,\ \mu>0$$

Provided integral exist.

3.7.1 ARA- Sawi Transform of some basic functions:

Sr. No.	$g(x,y)$	$\mathscr{G}x\,S_{y}\{g(x,y)\}=G(s,\mu)$
1	**1**	1
2	$x^a\,y^b$	$\dfrac{\mu^b\ a!\ \ b!}{s^a}$
3	e^{ax+by}	$\dfrac{s}{(s-a)\ (1-b\mu)}$
4	$e^{i(ax+by)}$	$\dfrac{is}{(s-ia)\ (b\mu+1)}$
5	$Sin(ax+by)$	$\dfrac{s(a+bs\mu)}{\left(a^2+s^2\right)\left(b^2\mu^2+1\right)}$
6	$Cos(ax+by)$	$\dfrac{s(s-ba\mu)}{\left(a^2+s^2\right)\left(b^2\mu^2+1\right)}$

7	$Sinh(ax+by)$	$\dfrac{s(a+bs\mu)}{(a^2-s^2)(b^2\mu^2-1)}$
8	$Cosh(ax+by)$	$\dfrac{s(s+ba\mu)}{(a^2-s^2)(b^2\mu^2-1)}$

3.7.2 Differential Property:

1. $\mathscr{G}_x S_y\{g_x(x,y)\} = s\,G(s,\mu) - s\,S\{g(0,y)\}$

2. $\mathscr{G}_x S_y\{g_y(x,y)\} = \dfrac{1}{\mu} G(s,\mu) - \dfrac{1}{\mu}\mathscr{G}\{g(x,0)\}$

3. $\mathscr{G}_x S_y\{g_{xx}(x,y)\} = s^2 G(s,\mu) - s^2 S\{g(0,y)\} - s\,S\{g_x(0,y)\}$

4. $\mathscr{G}_x S_y\left\{g_{yy}(x,y)\right\} = \dfrac{1}{\mu^2} G(s,\mu) - \dfrac{1}{\mu^2}\mathscr{G}\{g(x,0)\} - \dfrac{1}{\mu}\mathscr{G}\left\{g_y(x,0)\right\}$

5. $\mathscr{G}_x S_y\left\{g_{xy}(x,y)\right\} = \dfrac{s}{\mu} G(s,\mu) - S\{g(0,y)\} - \mathscr{G}\{g(x,0)\} - g(0,0)$

4. Key Theoretical and Behavioral Characteristics:

4.1 The Fourier -Laplace Transform:

Non symmetric kernel $e^{-i(st-ipx)}$ seamlessly splits steady spatial wave oscillations from decaying temporal dissipation.

4.2 The Laplace-ARA Transform:

Laplace ARA Transform combines classic exponential dampening with the ARA kernel se^{-st}. This combination offers superior boundary localization for singular partial derivatives.

4.3 The Sumudu-Sawi Transform:

Unit preserving scaling transform via dual non exponential kernels. It isolates parameter values directly without distorting the functions structural dimensions.

4.4 The Laplace- Sawi Transform:

Hybrid design leveraging Sawi's unique $\dfrac{1}{\varepsilon^2} e^{-\theta/\varepsilon}$ kernel. This significantly speeds up the conversation of volatile second order rates of change.

4.5 The ARA - Sawi Transform:

Highly focused algebraic simplification tool. It avoids complex integration by parts terms typically generated by standard Laplace method.

4.6 The ARA- Sumudu Transform:

Combines the powerful derivative handling feature of the ARA transform with the smooth, dimensionless scaling of the Sumudu-Transform.

4.7 The Sumudu Shehu Transform:

Uses a variable dependent Shehu Kernel $e^{-\frac{s}{u}t}$ to handle coupled systems of fractional equations that have non-local boundaries.

5. Special Advantages:

5.1 The Fourier -Laplace Transform:

- **Rigorous Phase shift and wave packets Trapping:** Unlike purely exponential transform, the spatial Fourier component uses a complex periodic kernel e^{-iwx} . This allows the transform to natively handle nonlocal spatial singular distributions.

- **Boundary De-coupling:** It perfectly mirrors wave mechanics spatial singularities moving symmetrically in both directions $(-\infty, \infty)$ are resolved as phase angles (ω) causal temporal evolution $(t \geq 0)$ is structurally bounded by the real damping parts of the Laplace parameter $(Re(s) > a)$.

5.2 The Laplace - ARA Transform:

The temporal ARA Transform introduces a flexible, indexing pre factor kernel (μ) that alters the standards exponential decay profile. In a distributional sense, this functions as a natural high-frequency filter. At the origin limit $(t \rightarrow 0)$, it cancels out structural poles and prevents the standard exponential boundary divergences that complicate classic multi-variable transform.

5.3 The Sumudu - Sawi Transform:

- Damping of High order singular explosions evaluating highly singular impulses such as the nth derivative of temporal singularity $\delta^{n}(t)$ introduces extreme polynomial divergence $\left(\nu^{-n}\right)$ into the frequency domain.

- The $\left(\nu^{2}\right)$ stabilizer: The Sawi kernels intrinsic $\frac{1}{\nu^{2}}$ pre factor acts as an aggressive analytical damper. It naturally absorbs and neutralizes two full orders of temporal differentiation spikes. This forces highly divergent, unstable generalized functions back into bounded uniformly convergent Sobolev-type algebraic spaces.

5.4 The Laplace- Sawi Transform:

Sawi transform introduces an inverted quadratic scaling parameter $\frac{1}{\mu^{2}}$ into the temporal domain integration. In distribution theory, evaluating high -order singular inputs (such as high order derivatives of the dirac delta function) triggers explosive polynomial growth. The invented quadratic factor behaves as an analytical damper aggressively neutralizing rapid polynomial or exponential growth and ensuring convergence within a strictly bounded algebraic domain.

5.5 The ARA- Sawi Transform:

Special Distributional advantages:

- Simultaneous pole-Cancelling and Growth Damping: This is an exceptionally robust architecture for handling singular partial differential equations with highly unstable initial boundaries.

- The Dual Parameter Shield $\left(\frac{p}{\nu^2}\right)$:The spatial ARA component introduces a linear multiplier P. In a distributional sense, this p-factor acts as a high-pass filter that cancels out structural poles and prevents co-ordinate singularities at the spatial origin boundary $(x \to 0)$.

Simultaneously, temporal Sawi component $\frac{1}{\nu^2}$ suppresses the explosive polynomial growth of temporal .shocks. Together they neutralize boundary divergences along both axes simultaneously

5.6 The ARA- Sumudu Transform:

This hybrid pairing completely eliminates unscaled exponential tracking by combining spatial indexing with the scale invariant properties of the Sumudu transform. Because the Sumudu parameter acts as an inverse scale element $\left(\frac{1}{\mu}\right)$ balanced by a spatial pre-factor (s) ,the resulting image preserves exact unit scale steps under derivative operations. This eliminates dimensional distortion when modeling generalized shock profiles in non- homogeneous mediums**.**

5.7 The Sumudu- Shehu Transform:

- Localized spatial-temporal mesh scaling: The Shehu kernel introduces a dynamic temporal shape metric (σ) ,while the Sumudu kernel preserves unit scale -invariance spatially. In distribution theory, this prevents the “smearing” of singular supports.
- Dynamic Shockwave Tracking: When a distribution represents a moving boundary front the (σ) parameter allows the domain of convergence to contract or expand dynamically along the time axis, matching the physical velocity of the singularity without altering the spatial test co-ordinates.

6. Applications:

6.1 The Fourier-Laplace Transform: Transient electrodynamics, acoustic wave guides, plasma physics and structural systems where spatial domains are completely open or infinite.

6.2 The Laplace-ARA Transform: Solving continuous partial differential equations like hyperbolic Klein-Gordon equation, structural wave equations and multivariable telegraph models.

6.3 The Sumudu-Sawi Transform: Integro-differential systems, multi-dimensional population dynamics and fluid flow boundary layers where scale invariance must be preserved.

6.4 The Laplace-Sawi Transform: High dimensional transport equations, non-uniform scaling problems and conformable fractional order system in dissipative physics**.**

6.5 The ARA-Sawi Transform: Boundary value problems containing fractional variable coefficient and signal transmission system with rapid physical dissipation**.**

6.6 The ARA-Sumudu Transform: Fractional Capito-derivative systems, Fokker-planks equations, and stochastic transport phenomena in science.

6.7 The Sumudu-Shehu Transform: Quantum mechanics wave mechanics, viscoelastic material modeling and fractional diffusion processes.

7. Conclusion:

In this paper, we have presented comparative study of the combined integral transforms such as Combined Fourier Laplace transform, combined Laplace ARA transform, Combined Laplace Sawi transform, combined Sumudu Sawi transform, combined Sumudu ARA transform, combined Sumudu Shehu transform, combined Sawi ARA transform through the derivation of transforms of basic functions, differential properties. Here we also discussed theoretical and behavioral properties of these transforms. Some distributional advantages of these transforms are also mentioned along with their applications in several forms.

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