Performing advance on inflation scenario description through Quadratic Teleparallel model

Abstract

The present paper constitutes a valuable contribution to the study of inflationary scenario through a comparison with Planck and BICEP2 observational data. Using a mathematical approach minimizing approximations in the theoretical description, the quadratic teleparallel model coupledwith a scalar field yields cosmological results consistent with observational investigations. Indeed, the free parameters of the model successfully reproduce the Planck and BICEP2 values ofthe spectral index and the tensor-to-scalar ration in the description of the inflationary scenario, leading to the following conclusions: the comoving Hubble radius decreases rapidly and approaches zero characterizing an exponential acceleration during the inflationary epoch; throughout the inflationary dynamic, the equation of state parameter remains negative, revealinga quintessence link evolution, while both the parameter and the scalar field potential decrease with time. Such behavior supports a graceful exit from inflation toward the reheating phase following a period of very rapid accelerated expansion.

Keywords: Inflation cosmology, Dark Energy, slow-roll, scalar factor, scalar field, teleparallel.

1. Introduction

One of the most intriguing questions in modern theoretical cosmology is whether the Universe originated from an initial singular state or whetherit Universe oscillates in a bouncing-like way, so the dilemma is, did the Universe started with a did the Universe start with a Big Bang or with a Big Bounce? Put another way, does the Universe evolves according to the standard inflationary paradigm, for which an initial singularity exists, or is the Universe described by a cosmological bounce, in which case the Universe has no initial singularity? This question is one of the most interesting questions in modern theoretical cosmology, and in order to be answered correctly, a lot of different theoretical disciplines have to be used, and must be supplemented by observational data. The successes of the standard inflationary scenario are quite many, for example the horizon and flatness problems are resolved, see for example [[1], [3]] for some review articles and important papers on this vast research topic. Inflation is a very particular phase of the early universe characterized by the fact that the expansion of the universe was accelerated exponentially for a very short period of time. We need to note that the inflationary scenario is a theoretical necessity in order to alleviate the problems of standard hot Big Bang cosmology[4]. It is the only coherent answer to the question of how the large-scale structure of matter was originally generated.

Several observational experiments support inflation, which is a purely theoretical concept without direct evidence. Very great experiments which try to track information on the origin of the universe can further give evidence for the inflation. From past to the future, we have: the Higgs discovery at the LHC [5] in 2012; different observations and detection of gravitational waves since 2015 [[6]-[7]] till 2023 in which year the NANOGrav [[8]-[9]] reported the first detection of a stochastic gravitational wave background; very significant progress is expected from future experiments, like the stage 4 Cosmic Microwave Background (CMB) experiments [10] and the future gravitational wave experiments like LISA and the Einstein Telescope [[11]-[14]]. Along with this previous scientific effort, there exist, in a quantitative way, direct experiments on the inflation scenario. Since 2013 [15], the Planck Satellite results not only confirm the basic principles of inflation but also provide the observable quantities which quantify the inflationary era. Over time, the Planck Satellite, the WMAP collaboration and the BICEP2 experiment [[16]-[20]] have dealt with the inflationary observable quantities. These observational data focus on the most important inflationary quantities, which are the spectral index of the scalar primordial curvature perturbations, the tensor spectral index of the primordial tensor perturbations and the tensor-to-scalar ratio. These data, which have become more refined since 2018, have always excited and guided research in the field of inflationary cosmology. The most spectacular example in terms of cosmological response to observational data is that of Andrei Linde after the Planck results of 2013 [2]. This interesting work gives specific emphasis to the new broad class of theories, the cosmological attractors, which have nearly model-independent predictions converging at the sweet spot of the Planck data in the plane of the spectral index of the scalar primordial curvature perturbations and the tensor-to-scalar ratio. In several theoretical descriptions, the inflationary observables are provided from others cosmological quantities call slow-roll parameters. Very recently, Prof Odintsov and his collaborators [4] performed interesting calculations of the slow-roll parameters for various mainstream theoretical frameworks and expressed the observational quantities of inflation in terms of the slow-roll indices needed for different theories. They review recent trends in inflationary dynamics in the context of viable modified gravity theories such as $f(R)$ theory, $f(R,\phi)$ theory, string motivated models, and Chern-Simons theories where R and $\phi$are the scalar curvature and the scalar field, respectively. Like these interesting works, we aim to analyze inflationary dynamics in a quantitative way for single scalar field inflation in another modified theory of gravity, the modified teleparallel theory of gravity.

The teleparallel theory is a standard theory of gravity like General Relativity. It is inspired bythe idea of constructing a theory bringing gravity closer to a gauge formulation and incorporating spin in a geometric description based on torsion. Later, it was shown to be equivalent to General Relativity in the description of gravitational interaction [21]. Then, real shortcomings arise when trying to explain the late time and the recent acceleration of the universe in comparison with observational data. This is the idea behind the modification of these theories [22]. In this context; the most popular theory in the teleparallel background is the $f(T)$ theory of gravity. Despite the impressive capabilities of this theory as compiled in [23], one question always comes back to researchers' minds: Can a universe driven by this type of gravity accommodate all cosmological and astronomical observations, and furthermore solve or avoid the theoretical problems existing in the standard inflationary ΛCDM paradigm? The present paper carries a simple and masterful technique of inflationary cosmology in order to test the viability of a theoretical $f(T)$ model, namely the quadratic model. This model is among the simplest inflationary models in the context of modified theories. It is inspired bythe most successful proposal in $f(R)$ by Starobinsky [24]. Recent interesting works like those in [25] and [26] have investigated some inflationary properties under the quadratic model. But here, our description is motivated by the reconstruction of observational data on the inflationary observables with satisfaction of exit from inflation.

The paper is organized as follows: in section, we present the field equations induced by the quadratic model and derivethe inflationary observables in section. In section, the cosmological evolution that confirms the inflationary scenario is undertaken and the conclusion concludes our workin section 5.

2. Motion equation induced by Quadratic teleparallel model

The teleparallel theory is one of the standard theories of gravitation used in cosmology and astrophysics. In the context of modifications, it gives rise to generalized versions among which the most basic one is, the $f(T)$ theory. Indeed, the modified teleparallel f(T) theory action is defined as [55].

$$S = \frac{1}{4k^2}\int d^4xhf(T) + \int d^4xhL_M \qquad (1)$$

where: T is the scalar torsion and represents the Einstein-Hilbert the teleparallel Lagrangian density,$h = |\det(h^a\mu)|$ is equivalent to $\sqrt{-g}$ in general relativity $k^2 = \frac{16\pi G}{C^4}, L_M$ is the Lagrange of thematter field. The motion equation is obtained from the variation of this action with respect to the tetrads $h^a{}_\mu$. One has

$$\frac{1}{h}\partial_\mu\left(hS_a^{\mu\nu}\right)f_T(T) - h_a^\lambda T^\rho \mu\lambda S_\rho^{\mu\nu} f_T(T) + A_{a\mu}^i S_i^{\mu\nu} f_T(T) + S_a^{\mu\nu}\partial_\mu(T)f_{TT}(T) + \frac{1}{4}h_a^\nu f(T) = \frac{1}{4k^2}T_a^\nu \quad (2)$$

where $f_T(T) = \frac{df(T)}{dT}, f_{TT}(T) = \frac{d^2f(T)}{dT^2}$ and T_a^νrepresents the energy-momentum tensor. We consider a universe powered by the Friedmann-Lemaitre-Robertson-Walker metric given by

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \qquad (3)$$

where $a(t)$ denotes the scale factor. The scalar torsion related to the metric Eq.(3) is given by

$$T = -6H^2(t),(4)$$

where $H(t)$ is the Hubble parameter. In the present work, we suppose that the universe is filled with perfect fluid powered by the scalar fieldϕ. In the context of Friedmannn-Lemaitre-Robertson-Walker metric, the appropriated form of the energy momentum tensor of perfect fluid is given by

$$T_{\mu\nu}= (\rho+ p)u_\mu u_\nu- pg_{\mu\nu}(5)$$

where $g_{\mu\nu}$and u_ν, are the metric tensor and the 4-vector characterizing a co-mobile observer, respectively. $\rho$and $p$are the global energy density and the pressure of universe content, respectively.Under these previous considerations, one can extract the Friedmannn-like equations of covariant modified Telleparallel theory

$$k^2\rho = 6H^2 f_T + \frac{1}{4}f \; and \; k^2\rho = 48\dot{H}H^2 f_{TT} - \left(2\dot{H} + 6H^2\right)f_T - \frac{1}{4}f \qquad (6)$$

In the present description, the only component of the universe content is supposed to be the scalar field. Indeed, the energymomentum tensor of the scalar field coming from the Noether theorem is given by

$$T_{\mu\nu} = \epsilon\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{\epsilon}{2}\partial_\beta\phi\partial^\beta\phi - V(\phi)\right] \qquad (7)$$

Here, $V(\phi)$ is the potential of the scalar field. By making using the previous metric, we deduce from (7), the energy-density and the pressure of the scalar field like several works such as [56]-[59].

$$\rho = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi)\ and\ p = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi) \qquad (8)$$

According to the previously cited work, theis constant. From cosmological work [54], ϵ= 1 corresponds to quintessence scalar scalar field whereas ϵ= -1 supports phantom evolution.

The system of equations traducing the interaction between the scalar field and the geometry in the framework of the modified theory are

$$k^2\left(\frac{\epsilon}{2}\dot{\phi}^2 + V(\phi)\right) = 6H^2 f_T + \frac{1}{4}f \qquad (9)$$

$$k^2\left(\frac{\epsilon}{2}\dot{\phi}^2 - V(\phi)\right) = 48\dot{H}H^2 f_{TT} - \left(2\dot{H} + 6H^2\right)f_T - \frac{1}{4}f \qquad (10)$$

The conservation equation $\dot{\rho} + 3H(\rho + p) = 0$, in the present context, leads to the following equation called Klein-Gordon equation [57]

$$\epsilon\ddot{\phi} + 3H\epsilon\dot{\phi} + V'(\phi) = 0 \qquad (11)$$

By adding the equations Eq.(9) to Eq.(11); we have

$$48\dot{H}H^2 f_{TT} - 2Hf_T = k^2\epsilon\dot{\phi}^2 \qquad (12)$$

The quadratic model that represents the guiding thread of this investigation is given by [25]

$$f(T) = T + \lambda T^2 \qquad (13)$$

Under the algebraic function Eq.(13), the motor equation Eq.(12) becomes

$$120\lambda\dot{H}H^2 = k^2\epsilon\dot{\phi}^2 \qquad (14)$$

Let us remark that the equation Eq.(**14**) is a differential equation of two separated e.fold number functions, $H(t)$ and $\phi(t)$. The function $H(t)$ is related to the space-time geometry whereas $\phi(t)$ is related to matter. The knowledge of these two functions is crucial, or simply indispensable, in an attempt to obtain the other cosmological quantities such as the scalar field potential and the inflationary observables. But here, to reach the goal, we have only one equation, the equation Eq.(14). The first idea can consist in choosing a cosmological ansatz expression for one of them and solving the equation Eq.(14) to find the second. For example, according to the literature (see [23]), it is possible to deduce the Hubble parameter *H*(*t*) describing the pawer-law expansion and the de sitter expansion. Such an approach is carried out in our recent work,[64]. Here, after analyzing the form of equation Eq.(14), we think about a simple mathematical way to solve it without supposing the Hubble parameter. Through such an approach, it is thereforepossible to provide an expression for the Hubble parameter in the description of inflationary scenario. Our approach to get two unknown functions from only one differential equation consists in introducing a constant *c* such that

$$120\lambda\dot{H}H^2 = k^2\epsilon\dot{\phi}^2 = c \qquad (15)$$

Such consideration limits approximation in theoretical description. After resolution of Eq.(16), one has

$$H(t) = \frac{\sqrt[3]{ct + 120\lambda s}}{2\sqrt[3]{5\sqrt[3]{\lambda}}} \qquad (16)$$

$$\phi(t) = \frac{\sqrt{ct}}{k\sqrt{\epsilon}} + \sigma \qquad (17)$$

Here *s* and *σ*are integration constants. We also have all ingredients to extract the scalar field potential from Klein-Gordon equation Eq.(12)

$$V(\phi) = \upsilon + \frac{9\left(\sqrt{ck}\sqrt{\epsilon}(\phi-\sigma)+120\lambda s\right)^{4/3}}{8\sqrt[3]{5}k^2\sqrt[3]{\lambda}} \qquad (18)$$

v is an integration constant. Like [54], in the following section, we will adopt quintessence evolution where$\epsilon = 1$.

2. Inflationary observable from quadratic model and comparison with observable

Recent trends in inflationary dynamics in the context of viable modified gravity theories [68] have revealed interesting motivation for dealing with inflation in *f*(*T*) theory. Furthermore, like many other works, a general review of inflationary cosmology and its present status, in view of the 2013 data release by the Planck satellite gives meaningful tools to confront theoretical descriptionswith observational predictions. This work tries to test a special inflationary model whose dynamics have been proved through several interesting works. For meaningful description of inflation, the exit from inflation is also taken into consideration in this investigation. So, we follow the same approach like [2]-[4]. To quantify the inflationary observables, one needs firstly the slow-roll parameters [2].

$$\epsilon = \frac{1}{2k^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \eta = \frac{1}{k^2}\left(\frac{V''(\phi)}{V(\phi)}\right) \qquad (19)$$

Basing on to the scalar field potential in Eq.(19), the slow-roll parameters become

$$\epsilon = \frac{360\sqrt[3]{5}c\left(\sqrt{c}k(\phi - \sigma) + 120\lambda s\right)^{2/3}}{\left(40k^2\sqrt[3]{\lambda}\upsilon - 95^{2/3}\left(\sqrt{c}k(\phi - \sigma) + 120\lambda s\right)^{4/3}\right)^2} \qquad (20)$$

$$\eta = -\frac{c}{2\sqrt[3]{5}k^2\sqrt[3]{\lambda}\left(\sqrt{c}k(\phi - \sigma) + 120\lambda s\right)^{2/3}\left(\upsilon - \frac{9\left(\sqrt{c}k(\phi-\sigma)+120\lambda s\right)^{4/3}}{8\sqrt[3]{5}k^2\sqrt[3]{\lambda}}\right)} \qquad (21)$$

Through the slow-roll parameters, it is possible to deal with the most popular observational quantities such as the spectral index of the primordial scalar curvature perturbations η_s and the tensor-to-scalar ratio r

$$\eta_s = 1 - 6\epsilon + 2\eta, \qquad r = 16\epsilon \qquad (22)$$

They can be expressed in term of scalar field ϕ as:

$$\eta_s = 1 - \frac{40\sqrt[3]{5}c\left(45\left(\sqrt{c}k(\phi-\sigma)+120\lambda s\right)^{4/3}+8\sqrt[3]{5}k^2\sqrt[3]{\lambda}\upsilon\right)}{\left(\sqrt{c}k(\phi-\sigma)+120\lambda s\right)^{2/3}\left(40k^2\sqrt[3]{\lambda}\upsilon-95^{2/3}\left(\sqrt{c}k(\phi-\sigma)+120\lambda s\right)^{4/3}\right)^2} \qquad (23)$$

$$r = \frac{5760\sqrt[3]{5}c\left(\sqrt{c}k(\phi - \sigma) + 120\lambda s\right)^{2/3}}{\left(40k^2\sqrt[3]{\lambda}\upsilon - 95^{2/3}\left(\sqrt{c}k(\phi - \sigma) + 120\lambda s\right)^{4/3}\right)^2} \qquad (24)$$

At the end of inflation, the slow-roll parameters behave like $\epsilon \simeq 1$. This condition leads to the possibility of finding the scalar field ϕ_f corresponding to the end of inflation. But some shortcoming arises when solving the associated equation. To overcome this, like in [2], we suppose that, at the end of inflation the scalar field becomes very low such that the first slow-roll parameter can be written as:

$$\epsilon \simeq \frac{360\sqrt[3]{5}cA^{2/3}\left(1+\frac{2k\sqrt{c}}{3A}\phi f\right)}{\left[40k^2\sqrt[3]{\lambda}v - 95^{2/3}A^{4/3}\left(1+\frac{4k\sqrt{c}}{3A}\phi f\right)\right]^2} \simeq 1 \qquad (25)$$

where $A = (120s\lambda - \kappa\sigma\sqrt{c}$. The scalar field at the end of inflation is extracted as

$$\phi f = \frac{40\sqrt[3]{5}A^{2/3}\sqrt{c}k^3\sqrt[3]{\lambda}v - 45A^2\sqrt{c}k + 25^{2/3}\sqrt[3]{A}\sqrt{c^2k^2\left(\frac{5^{2/3}(9A^2+c)}{A^{2/3}} + 40k^2\sqrt[3]{\lambda}v\right)} + 10c^{3/2}k}{60Ack^2} \qquad (26)$$

When addressing the inflationary scenario, the e.folds number is used to represent the time between two epochs. In the present case, it covers the inflationary era (from beginning to exit from inflation). From its definition [4], it is possible to express the e.folds number in terms of the scalar field by using the equations Eq.(16) and Eq.(17). One has:

$$N = \int_t^{t_f} H(t)dt = \int_\phi^{\phi_f} \frac{k\left[k\sqrt{c}(\phi - \sigma) + 120s\sigma\right]^{1/3}}{\sqrt[2]{c}(5\lambda)^{1/3}} d\phi \qquad (27)$$

We compute the relation Eq.(27) and express the scalar field as e.folds number function by using the relation Eq.(26). One has:

$$\phi(N) = \frac{\sqrt{3}\sqrt[4]{5}\left[\frac{10(4\sqrt[3]{5}A^{2/3}k^2\sqrt[3]{\lambda}v - 12\sqrt[3]{5}Ac\sqrt[3]{\lambda}N + c)}{A} + \frac{25^{2/3}\sqrt{c^2k^2\left(\frac{5^{2/3}(9A^2+c)}{A^{2/3}} + 40k^2\sqrt[3]{\lambda}v\right)}}{A^{2/3}\sqrt{c}k}\right]^{3/4} - 45A}{45\sqrt{c}k} \qquad (28)$$

The inflationary observables in Eq.(23) and Eq.(24) become

$$r = \frac{965^{5/6}A^2c^2k^2\sqrt{\frac{10(4\sqrt[3]{5}A^{2/3}k^2\sqrt[3]{\lambda}v - 12\sqrt[3]{5}Ac\sqrt[3]{\lambda}N + c)}{A} + \frac{25^{2/3}\sqrt{c^2k^2\left(\frac{5^{2/3}(9A^2+c)}{A^{2/3}} + 40k^2\sqrt[3]{\lambda}v\right)}}{A^{2/3}\sqrt{c}k}}}{\left[\sqrt{c}k\left(c\left(5^{2/3} - 60A\sqrt[3]{\lambda}N\right) - 20\left(\sqrt[3]{A} - 1\right)A^{2/3}k^2\sqrt[3]{\lambda}v\right) + \sqrt[3]{5}\sqrt[3]{A}\sqrt{c^2k^2\left(\frac{5^{2/3}(9A^2+c)}{A^{2/3}} + 40k^2\sqrt[3]{\lambda}v\right)}\right]^2} \qquad (29)$$

$$\eta_s = 1 - \frac{\Xi(N)}{\chi(N)}$$

With

$$\Xi(N) = 150\sqrt{2}\sqrt[6]{5}Ac^{3/2}k\left[\sqrt{c}k\left(4A^{2/3}(\sqrt[3]{A}+5)k^2\sqrt[3]{\lambda}v + c\left(5^{2/3} - 60A\sqrt[3]{\lambda}N\right)\right) + \sqrt[3]{5}\sqrt[3]{A}\sqrt{c^2k^2\left(\frac{5^{2/3}(9A^2+c)}{A^{2/3}} + 40k^2\sqrt[3]{\lambda}v\right)}\right] \qquad (30)$$

$$\chi(N) = \left[\sqrt{c}k\left(c\left(5^{2/3} - 60A\sqrt[3]{\lambda}N\right) - 20\left(\sqrt[3]{A} - 1\right)A^{2/3}k^2\sqrt[3]{\lambda}v\right) + \sqrt[3]{5}\sqrt[3]{\boldsymbol{A}}\sqrt{c^2k^2\left(\frac{5^{2/3}(9A^2+c)}{A^{2/3}} + 40k^2\sqrt[3]{\lambda}v\right)}\right]^2 \sqrt{\frac{5^{2/3}\sqrt{c^2k^2\left(\frac{5^{2/3}(9A^2+c)}{A^{2/3}} + 40k^2\sqrt[3]{\lambda}v\right)}}{A^{2/3}\sqrt{c}k} + \frac{5c}{A} + \frac{20\sqrt[3]{5}k^2\sqrt[3]{\lambda}v}{\sqrt[3]{A}} - 60\sqrt[3]{5}c\sqrt[3]{\boldsymbol{\lambda}}N} \quad (31)$$

Several observational data are collected on these observables. We present here some recent observational results on the spectral index n_s, the tensor-to-scalar ratio r and the running of the spectral index α_s. The recent data from the Planck satellite [31] suggested $\eta_s = 0.9603 \pm 0.0073(68\%CL)$,$r<0.11(95\%CL)$, and $\alpha_s = -0.0134 \pm 0.0090(68\%CL)$ [Planck et WMAP ([18]; [17]], whose negative sign is at 1.5σ. Furthermore, Planck 2018 TT,TE,EE+lowEB+lensing [66] suggests $\eta_s = 0.9659 \pm 0.0041(68\%CL)$. The BICEP2 experiment [31] implies $r = 0.20^{+0.07}_{-0.05}(68\%CL)$. It is mentioned that discussions exist on how to subtract the foreground, for example in [31],[16]. Recently, progress has appeared also in [19] to ensure the BICEP2 declarations. It has also been remarked that the representation of α_s is presented in [20].

The contribution of the quadratic model in the theoretical description of these observables is clearly shown through Fig.1 to Fig.3. Indeed, Fig.1 reveals that through the quadratic model, the tensor-to-scalar ratio r typically decreases with the increasing of the e.fold number N. Such evolution confirms several observational data on the tensor-to-scalar ratio r. From this figure one has $r < 0.12$. Fig.2 shows the spectral index evolution versus the e.folds number whereas the parametric plot in Fig.2 involves the correlation between the two observables. Especially in the present investigation, we provide in each figure the band corresponding to the Planck 2018 TT,TE,EE+lowEB+lensing [66]. This illustration shows clearly that the quadratic model can reproduce the inflationary scenario of the universe in accordance with observational data. It is also clear from Fig.1 to Fig.3 that a good part of each curve lies in the Planck data band. The end of inflation corresponding to N = 60 is also covered by the band in each figure. The variations of the spectral index and the tensor-to-scalar ratio presented in Fig.1 to Fig.3 fit the values predicted by [20] and [65]. These conclusions reflect the consistency of this theoretical description with the observational data.

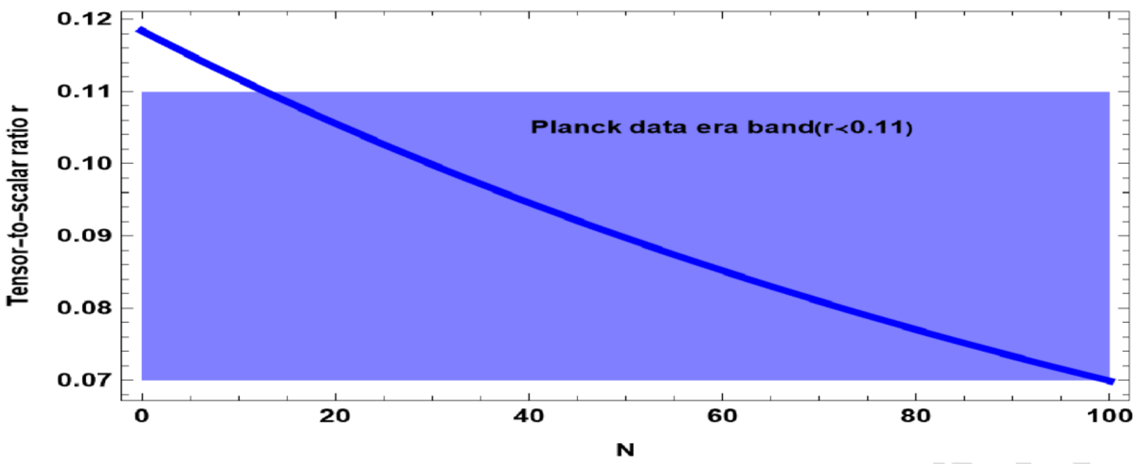

Figure 1: Evolution versus e.fold Number N of tensor-to-scalar ratio r in quadratic model background. The blue curve traducing the variation of the tensor-to-scalar ratio with e.folds number is obtained for $c = 0.01$; $s = 20$; $\upsilon = 2.5.10^{27}$; $\lambda = 0.001$; $\sigma = -25$; $k \sqrt{\frac{1.8626}{10^{26}}}$ *SI*

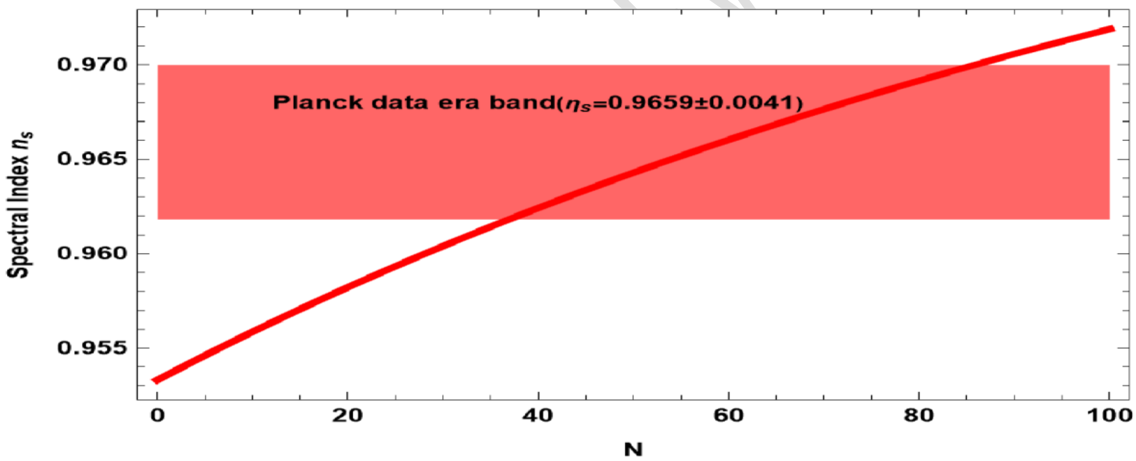

Figure 2:Evolution versus e.fold Number N of the spectral index η_s *in quadratic model background. The red curve traducing the variation of the spectral index with e.folds number is obtained for* $c = 0.01$; $s = 20$; $\upsilon = 2.5.10^{27}$; $\lambda = 0.001$; $\sigma = -25$; $k \sqrt{\frac{1.8626}{10^{26}}}$ *SI*

4. Cosmological evolution powered by the inflationary picture

4.1. Test with comoving Hubble radius

In this section, we deal with an important cosmological quantity to support the inflationary scenario powered by our previous description. Here, attention will be paid to the comoving Hubble radius. It strongly depends on the Hubble parameter which constitutes one of the main research results of this investigation. It was found in [4] that the fundamental condition for inflation to occur is defined by the negative time derivative of the comoving Hubble radius whereas the end of inflation occurs when the comoving Hubble radius stops decreasing. Such results are also supported by [23] and [62]. According to the previously cited sources, the comoving Hubble radius can be expressed as:

$$R_H(t) = (aH)^{-1} \;\rightarrow\; R_H(t) = \left(a_0 H e^{\int H dt}\right)^{-1} \qquad (32)$$

where a_0 is an integration constant. From our result in (17), the Hubble radius becomes

$$R_H(t) = \frac{2\sqrt[3]{5}\sqrt[3]{\lambda}}{a_0\sqrt[3]{ct+120\lambda s}\,\exp\left(\frac{3(ct+120\lambda s)^{4/3}}{8\sqrt[3]{5}c\sqrt[3]{\lambda}}\right)} \qquad (33)$$

In the present situation where the results agree with observational data, we depict the comoving Hubble radius versus cosmic time and obtain the curve in Fig.4. From the figure Fig.4, it follows that under the conditions for which the inflationary observables mimic the observational data, the same conditions imply a Hubble radius which quickly decreases with cosmic time. Furthermore, the evolution presented in this figure seems to provide Hubble radius vanishing at the moment the inflation

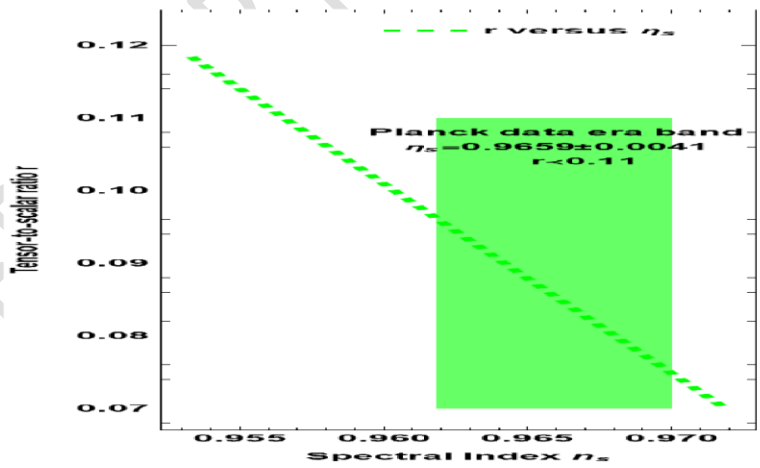

Figure 3: Parametric plot of tensor-to-scalar ratio r versus spectral index η_s in quadratic model background. The green curve traducing the variation of the tensor-to-scalar ratio with the spectral index is obtained for c = 0.01; s = 20; υ = 2.5.10^{27}; λ = 0.001; σ = -25; k $\sqrt{\frac{1.8626}{10^{26}}}$ SI

ends. Such conclusion is also carried out in [4]. To be convinced ofsuch a feature, we express the comoving Hubble radius with respect to the e.fold number under the scenario described in the previous section where the end from inflation is strictly taken into consideration. This consists in expressing the comoving Hubble radius as e.fold number before depicting it. So, by using the relations Eqs.(18),(29),(34), one has

$$R_H(N) = \frac{2^{3/4}\sqrt{3}5^{7/12}\exp\left(\frac{\sqrt[3]{\lambda}\Delta(N)}{60c^{3/2}k\sqrt[3]{\lambda}(\sqrt{c}k\sigma-120\lambda s)}\right)}{\Gamma(N)} \quad (34)$$

With

$$\Delta(N) = 5^{2/3}c^{3/2}k - 7200c^{3/2}k\lambda^{4/3}Ns + 60c^2k^2\sqrt[3]{\lambda}N\sigma + \sqrt[3]{600\lambda s - 5\sqrt{c}k\sigma}\sqrt{c^2k^2\left(\frac{9(\sqrt{c}k\sigma-120\lambda s)^2+c}{\left(24\lambda s-\frac{1}{5}\sqrt{c}k\sigma\right)^{2/3}}+40k^2\sqrt[3]{\lambda}\upsilon\right)} + 20\sqrt{c}k^3\sqrt[3]{\lambda}\upsilon(120\lambda s - \sqrt{c}k\sigma)^{2/3}$$

$$\Gamma(N) = d\left(\frac{-\Upsilon(N)}{ck^2\sigma - 120\sqrt{c}k\lambda s}\right)^{1/4}$$

$$\Upsilon(N) = 5c^{3/2}k - 7200\sqrt[3]{5}c^{3/2}k\lambda^{4/3}Ns + 60\sqrt[3]{5}c^2k^2\sqrt[3]{\lambda}N\sigma + 20\sqrt[3]{5}\sqrt{c}k^3\sqrt[3]{\lambda}\upsilon(120\lambda s - \sqrt{c}k\sigma)^{2/3} + 5^{2/3}\sqrt[3]{120\lambda s - \sqrt{c}k\sigma}\sqrt{c^2k^2\left(\frac{9(\sqrt{c}k\sigma-120\lambda s)^2+c}{\left(24\lambda s-\frac{1}{5}\sqrt{c}k\sigma\right)^{2/3}}+40k^2\sqrt[3]{\lambda}\upsilon\right)} \quad (35)$$

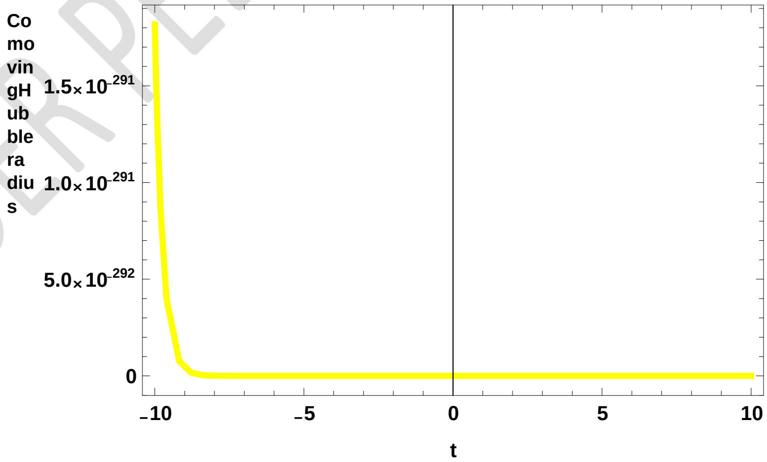

Figure 4:Comoving Hubble radius evolution versus cosmic time in quadratic model background.

The curve is obtained for $c = 0.01; s = 20; \upsilon = 2.5.10^{27}; \lambda = 0.001; \sigma = -25; k \sqrt{\frac{1.8626}{10^{26}}}$ *SI*

The relation in Eq.(35) shows that the comoving Hubble radius depends stronglyon the e.fold number in the context ensuring the end from inflation. Fig.5, shows the evolution of the Hubble radius versus e.fold number during inflation with possible exist. This figure indicates that, although there is ane.fold dependence of the Hubble radius, it is practically equal to zero during the inflation meaning that the scalar factor grows very quickly during the inflation. Such conclusion is also found in [4]. Furthermore, such a Hubble radius acts as a universal isolator,freezing quantum fluctuations to form the building blocks of our current universe while simultaneously separating space into autonomously evolving domains.

4.2. Cosmological evolution during inflation

Cosmological evolution during inflation also remains a viable indicator to test a theoretical inflationary model. Several inflationary models have been submitted to thiscriterion which gives more cosmological scope to theoretical description [23]. To achievethis goal, we aim to provide the state equation parameter of the scalar field in terms of e.fold number. First of all, we come back to the expression of the scalar field potential of Eq.(19), and to the energy density and the pressure of the scalar field in Eq.(8). During the inflationary epoch, their variations are powered by the following expressions:

$$V(N) = \frac{1}{20\sqrt{c}k^2\sqrt[3]{\lambda}(\sqrt{c}k\sigma - 120\lambda s)}\Bigg[2400\sqrt{c}k\lambda^{4/3}s(3cN - k^2 \upsilon) - 20\sqrt{c}k^3\sqrt[3]{\lambda}\upsilon\left(120\lambda s - \sqrt{c}k\sigma\right)^{2/3}ck\left(60ck\sqrt[3]{\lambda}N\sigma + 5^{2/3}\sqrt{c} - 20k^3\sqrt[3]{\lambda}\sigma\upsilon\right) - \sqrt[3]{600\lambda s - 5\sqrt{c}k\sigma}\sqrt{c^2k^2\left(\frac{9\left(\sqrt{c}k\sigma - 120\lambda s\right)^2 + c}{\left(24\lambda s - \frac{1}{5}\sqrt{c}k\sigma\right)^{2/3}} + 40k^2\sqrt[3]{\lambda}\upsilon\right)}\Bigg] \tag{36}$$

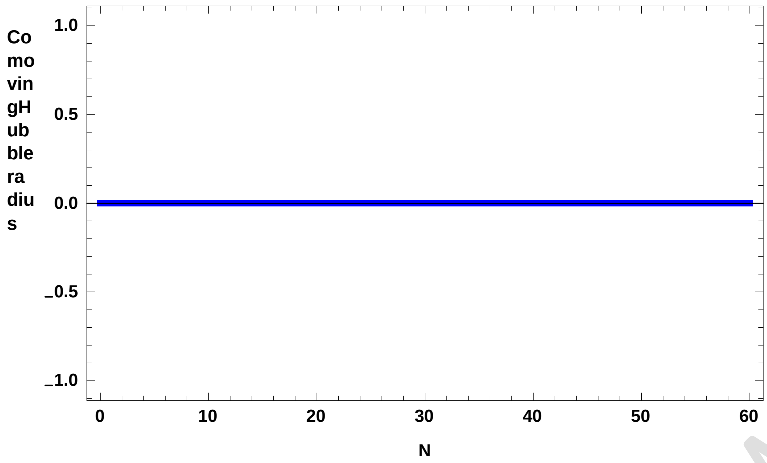

Figure 5:Comoving Hubble radius evolution versus cosmic time in quadratic model background.

The curve is obtained forc = 0.01; s = 20; υ = 2.5.10^{27}*; λ = 0.001; σ = -25; k* $\sqrt{\frac{1.8626}{10^{26}}}$ *SI*

$$\rho(N) = \frac{1}{20\sqrt{c}k^3\sqrt[3]{\lambda}(\sqrt{c}k\sigma-120\lambda s)}\Bigg[2400\sqrt{c}k\lambda^{4/3}s(3cN - k^2\upsilon) - 20\sqrt{c}k^3\sqrt[3]{\lambda}\upsilon(120\lambda s - \sqrt{c}k\sigma)^{2/3} - ck\Big(60ck\sqrt[3]{\lambda}N\sigma +$$

$$5^{2/3}\sqrt{c} - 20k^3\sqrt[3]{\lambda}\sigma\upsilon\Big) - \sqrt[3]{600\lambda s - 5\sqrt{c}k\sigma}\sqrt{c^2k^2\left(\frac{9(\sqrt{c}k\sigma-120\lambda s)^2+c}{\left(24\lambda s-\frac{1}{5}\sqrt{c}k\sigma\right)^{2/3}} + 40k^2\sqrt[3]{\lambda}\upsilon\right)}\Bigg] + \frac{1}{4050\,ck^2}\Bigg[45\sqrt{c}k\sigma - 5400\lambda s +$$

$$2^{3/4}\sqrt[4]{5}\sqrt{3}\left(\frac{5c}{120\lambda s-\sqrt{c}k\sigma} - 60\sqrt[3]{5}c\sqrt[3]{\lambda}N + \frac{20\sqrt[3]{5}k^2\sqrt[3]{\lambda}\upsilon}{\sqrt[3]{120\lambda s-\sqrt{c}k\sigma}} + \frac{\sqrt{c^2k^2\left(\frac{9(\sqrt{c}k\sigma-120\lambda s)^2+c}{\left(24\lambda s-\frac{1}{5}\sqrt{c}k\sigma\right)^{2/3}}+40k^2\sqrt[3]{\lambda}\upsilon\right)}}{\sqrt{c}k\left(24\lambda s-\frac{1}{5}\sqrt{c}k\sigma\right)^{2/3}}\right)^{3/4}\Bigg]^2 \quad (37)$$

$$p(N) = \frac{1}{20\sqrt{c}k^3\sqrt[3]{\lambda}(\sqrt{c}k\sigma - 120\lambda s)}\left[2400\sqrt{c}k\lambda^{4/3}s(3cN - k^2 v) + 20\sqrt{c}k^3\sqrt[3]{\lambda}v(120\lambda s - \sqrt{c}k\sigma)^{2/3} + ck\left(60ck\sqrt[3]{\lambda}N\sigma + 5^{2/3}\sqrt{c} - 20k^3\sqrt[3]{\lambda}\sigma v\right) + \sqrt[3]{600\lambda s - 5\sqrt{c}k\sigma}\sqrt{c^2k^2\left(\frac{9(\sqrt{c}k\sigma - 120\lambda s)^2 + c}{\left(24\lambda s - \frac{1}{5}\sqrt{c}k\sigma\right)^{2/3}} + 40k^2\sqrt[3]{\lambda}v\right)}\right] + \frac{1}{4050ck^2}\left[45\sqrt{c}k\sigma - 5400\lambda s + 2^{3/4}\sqrt[4]{5}\sqrt{3}\left(\frac{5c}{120\lambda s - \sqrt{c}k\sigma} - 60\sqrt[3]{5}c\sqrt[3]{\lambda}N + \frac{20\sqrt[3]{5}k^2\sqrt[3]{\lambda}v}{\sqrt[3]{120\lambda s - \sqrt{c}k\sigma}} + \frac{\sqrt{c^2k^2\left(\frac{9(\sqrt{c}k\sigma - 120\lambda s)^2 + c}{\left(24\lambda s - \frac{1}{5}\sqrt{c}k\sigma\right)^{2/3}} + 40k^2\sqrt[3]{\lambda}v\right)}}{\sqrt{c}k\left(24\lambda s - \frac{1}{5}\sqrt{c}k\sigma\right)^{2/3}}\right)^{3/4}\right]^2 \quad (38)$$

The last two equations must lead to the expression equation parameter expression as a function of the e.fold number *ω*(*N*) = *p*(*N*)/*ρ*(*N*). Fig.6 presents the evolution of the both scalar field potential and the state equation parameter in a context where the free parameters used in this investigation have led to Planck and BICEP2 experiment results. Interesting cosmological features arise from the variation presented in Fig.6. First, the scalar field potential decreases during the inflationary ear. Such a result supports the idea that the scalar field loses its energy during inflation in order to allow for the formation of structures in the universe. In other words, under this decrease, it is possible to have a minimum potential matching the beginning of elementary particles creation and the reheating process. Such conclusion is also defended in [2] and [63]. Secondly, in the same figure, it is shown that

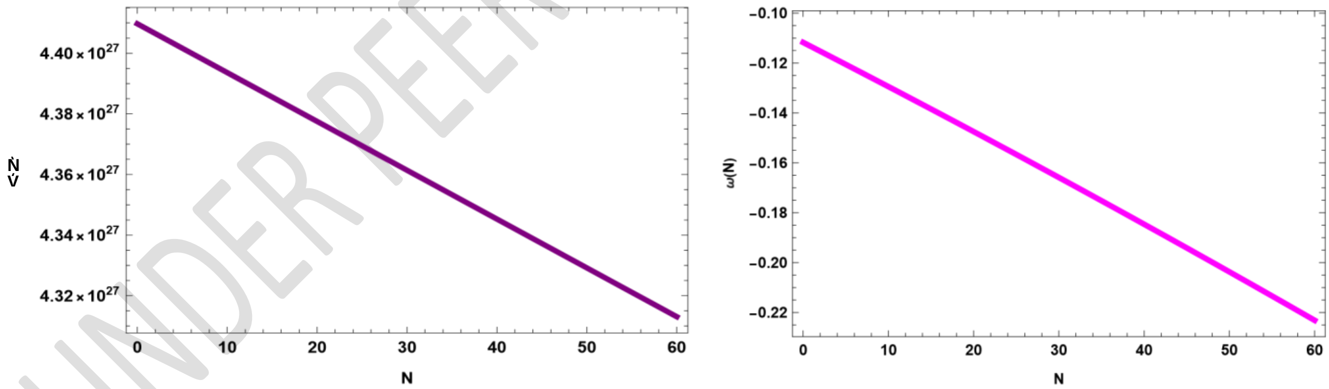

Figure 6: Evolution of the scalar field potential (purple curve at left) and the state equation parameter (magenta curve at right) versus the e.fold number during the inflationary era. The curvesare obtained forc = 0.01; s = 20; υ = 2.5.10^{27}*; λ = 0.001; σ = -25; k* $\sqrt{\frac{1.8626}{10^{26}}}$ *SI*

the equation of state parameter is negative and decreases during the inflationary era. This means that during inflation, the pressure is negative and its absolute value increases. In addition to the behavior of the Hubble radius in Fig.4 and Fig.5, this behavior of the equation of state parameter confirms the very accelerated expansion during inflation. Furthermore, the right curve of Fig.5 showing the variation of the state equation parameter shows$-\frac{1}{3} < \omega > 0$. So, the scalar field promoted by the quadratic teleparallel model demonstrates quintessence model features during the inflationary scenario. Such a scalar field is confirmed by a number of cosmological measurements, such as the CMB Radiation, LSS formation, and type Ia supernovae observations [71]-[75] for the simple reason that it begins in the quintessence region $-\frac{1}{3} < \omega > 0$and probably tends towards the special value $\omega = -1$ characterizing the ΛCDM model. The interesting results provided via Fig.5 on the equation of state parameter reinforce and confirm the choice of a quintessence scalar field made in the first section in light of the work [54].

5. Conclusion

The present investigation is devoted to the cosmological implication of the quadratic teleparallel model in the inflationary scenario description by means of very advanced tools on this topic. After providing the main equation in modified teleparallel theory, the motion equation induced by the quadratic model in the presence of a scalar field is deriveed. This master equation gives the possibility of being solved by a simple mathematical method leading to the scalar field and the Hubble parameter expressions. By solving the Klein-Gordon equation, we provide the scalar field potential which represents the most important ingredient in this description. Our goal of investigating the inflationary scenario in a description where observational data are fitted and the exit from inflation is ensured, is defended through two sections.

Firstly, the inflationary scenario is addressed by the introduction of the slow-roll parameters and the observables. Like several works in the literature, the condition for exit from inflation and the introduction of the e.fold number have made possible the theoretical construction of the observables before testing them with Planck and other observational data. Fig.1 to Fig.3 show that under suitable values of our free parameters, the spectral index and the tensor-to-scalar ratio fit the observational data. What is the cosmological evolution behind the inflationary picture promoted by our model?

Secondly, during the inflationary dynamics, the cosmological evolution powered by the quadratic teleparallel model in a universe filled with a quintessential scalar field is tested by defining some quantities which are important for a better understanding of the concept of inflation. The comoving Hubble radius is investigated in both analytical and numerical ways. The numerical analysis provides in Fig.3 and Fig.4 states that the comoving Hubble radius decreases quickly as comic time flows by. Thus, it is practically zero during the inflationary period, thus reflecting a kind of very rapid acceleration during this phase. Furthermore, the scalar field considered in this work is quintessence-like. We provide in this work its potential, energy density, pressure and the equation of state parameter. The numerical analysis of the scalar field potential via Fig.5 demonstrates a decreasing scalar potential, proving that the inflationary exit is guaranteed and can lead to the reheating era. In the same figure, the equation of state parameter reflects the accelerated expansion in the quintessential scenario which not only confirms the nature of the scalar field but also the inflationary era. These results are numerically illustrated with the free parameter values under which the inflationary observables induced by the quadratic model fit the observational data from the Planck satellite and other observations.

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