



32  $p$  does not divide  $-1$ . Here, we extend the results of Amita Sahni and P.T.  
 33 Sehgal [5].

34 **REMARK2.1** For  $0 \leq s \leq z - 1$ , let  $C_s = \{s, sl, sl^2, \dots, sl^{t_s-1}\}$ , where  $t_s$  is  
 35 the least positive integer such that  $sl^{t_s} \equiv s \pmod{8p^n q^m}$  be the cyclomic  
 36 coset containing  $s$ .

37 **LEMMA2.1.** Let  $p, q, l$  be distinct odd primes,  $n \geq 1$  an  
 38 integer,  $o(l)_{8p^{n-j}} = \varphi(8p^{n-j}), o(l)_{q^{m-k}} = \varphi(q^{m-k})$  and  
 39  $gcd(\varphi(8p^{n-j}), \varphi(q^{m-k})) = d$  then  $o(l)_{8p^{n-j}q^{m-k}} = \frac{\varphi(8p^{n-j}q^{m-k})}{d}$ , for  
 40 all  $0 \leq j \leq n - 1$  and  $0 \leq k \leq m - 1$ .

41 **Proof.** Let  $o(l)_{8p^{n-j}q^{m-k}} = t, 0 \leq j \leq n - 1$  and  $0 \leq k \leq m - 1$ . Then  
 42  $l^t \equiv 1 \pmod{8p^{n-j}q^{m-k}}$  But  $p$  and  $q$  are distinct odd primes. Hence  
 43  $l^t \equiv 1 \pmod{8p^{n-j}}$  and  $l^t \equiv 1 \pmod{q^{m-k}}$ . Since  $o(l)_{8p^{n-j}} = \varphi(8p^{n-j})$   
 44 and,  $o(l)_{q^{m-k}} = \varphi(q^{m-k})$  therefore,  $\varphi(8p^{n-j})$  and  $\varphi(q^{m-k})$  divides  
 45  $t$ . Then,  $lcm(\varphi(8p^{n-j}), \varphi(q^{m-k})) = \frac{\varphi(8p^{n-j}q^{m-k})}{d}$  divides  $t$ . On the other  
 46 hand, since  $o(l)_{q^{m-k}} = \varphi(q^{m-k})$ , therefore,  $l^{\varphi(q^{m-k})} \equiv 1 \pmod{q^{m-k}}$   
 47 hence  $l^{\varphi(\frac{8p^{n-j}q^{m-k}}{d})} \equiv 1 \pmod{q^{m-k}}$ . Similarly,  $l^{\varphi(\frac{8p^{n-j}q^{m-k}}{d})} \equiv$   
 48  $1 \pmod{8p^{n-j}}$ . As  $p$  and  $q$  are distinct primes, therefore, we get  
 49  $l^{\varphi(\frac{2p^{n-j}q^{m-k}}{d})} \equiv 1 \pmod{8p^{n-j}q^{m-k}}$

50 Hence,  $t = o(l)_{8p^{n-j}q^{m-k}}$  divides  $\frac{\varphi(8p^{n-j}q^{m-k})}{d}$  and we get that  
 51  $t = \frac{\varphi(8p^{n-j}q^{m-k})}{d}$ .

52 **LEMMA2.2.** For given  $p, q, l$  distinct odd primes such that  $g.c.d.$   
 53  $(\varphi(p), \varphi(q)) = d$ , and  $l$  is a primitive root mod  $(p)$  as well as  $q$ , then there  
 54 always exists a fixed integer  $a$  satisfying  $g.c.d.(a, pq) = 1, 1 < a < pq$ , such  
 55 that  $a$  is a primitive root mod  $(p)$  and the order of  $a$  mod  $q$  is  $\varphi(q)$ . Also  
 56  $a, a^2, a^3, \dots, a^{d-1}$  does not belong to the set  $S = \{1, l, l^2, \dots, l^{\frac{\varphi(pq)}{d}-1}\}$ . Further,  
 57 for this fixed integer  $a$  and for  $0 \leq j \leq n - 1, 0 \leq k \leq m - 1$  the set  $\{1, l, l^2,$   
 58  $\dots, l^{\frac{\varphi(8p^{n-j}q^{m-k})}{d}-1}, a, al, \dots, al^{\frac{\varphi(8p^{n-j}q^{m-k})}{d}-1}, a^2, a^2l, a^2l^2, \dots,$

59  $a^2 l^{\frac{\varphi(8p^{n-j}q^{m-k})}{d}-1} a^{d-1}, a^{d-1}l, \dots, a^{d-1}l^{\frac{\varphi(8p^{n-j}q^{m-k})}{d}-1}$  forms a reduced residue  
60 system modulo  $8p^{n-j}q^{m-k}$ .

61 **Proof:** Similar Lemma 4 [10]

62 **THEOREM2.1.** If  $\eta = 8p^n q^m$  ( $m, n \geq 1$ ), Then the  $8(m \times n \times d + m +$   
63  $n + 1)$  Cyclotomic cosets modulo  $8p^n q^m$  are given by (i)  $C_0 = \{0\}$   
64  $, (ii) C_{p^n q^m} = \{p^n q^m\}, (iii) C_{2p^n q^m} = \{2p^n q^m\}, (iv) C_{3p^n q^m} = \{3p^n q^m$   
65  $\}, (v) C_{4p^n q^m} = \{4p^n q^m\}, (vi) C_{5p^n q^m} = \{5p^n q^m\}, (vii) C_{6p^n q^m} = \{6p^n q^m\}$   
66  $, (viii) C_{7p^n q^m} = \{7p^n q^m\}$ . For  $0 \leq k \leq m-1$

67  $C_{p^n} = \{p^n, p^n l, \dots, p^n l^{\varphi(q^{m-k})-1}\}, (iv) C_{2p^n} = \{2p^n, 2p^n l, \dots, 2p^n l^{\varphi(q^{m-k})-1}\}$   
68  $, (v) C_{3p^n} = \{3p^n, 3p^n l, \dots, 3p^n l^{\varphi(q^{m-k})-1}\} \dots (vi) C_{4p^n} = \{4p^n, 4p^n l, \dots,$   
69  $4p^n l^{\varphi(q^{m-k})-1}\} \dots (vii) C_{5p^n} = \{5p^n, 5p^n l, \dots, 5p^n l^{\varphi(q^{m-k})-1}\}$   
70  $\dots (viii) C_{6p^n} = \{6p^n, 6p^n l, \dots, 6p^n l^{\varphi(q^{m-k})-1}\} \dots (ix) C_{7p^n} = \{7p^n, 7p^n l, \dots,$   
71  $7p^n l^{\varphi(q^{m-k})-1}\}$  and for  $0 \leq j \leq n-1$ ,

72  $(i) C_{q^m} = \{q^m, q^m l, \dots, q^m l^{\varphi(p^{n-j})-1}\}$

73  $(ii) C_{2q^m} = \{2q^m, 2q^m l, \dots, 2q^m l^{\varphi(p^{n-j})-1}\}$  Similarly we can obtain  
74 remaining 6 Cyclotomic cosets.

75 For  $0 \leq j \leq n-1$ , and  $0 \leq k, \leq m-1$  for  $0 \leq w \leq d-1$ ,

76  $(i) C_{a^w p^j q^k} = \{ a^w p^j q^k, a^w p^j q^k l, \dots, a^w p^j q^k l^{\frac{\varphi(2p^{n-j}q^{m-k})}{d}-1} \}$

77  $, (ii) C_{2a^w p^j q^k} = \{ 2a^w p^j q^k, 2a^w p^j q^k l, \dots, 2a^w p^j q^k l^{\frac{\varphi(2p^{n-j}q^{m-k})}{d}-1} \}$

78 where the number  $a$  is given by Lemma 4[10] Similarly we can  
79 obtain remaining 6 Cyclotomic cosets.

80 .

81 **Proof:-** As per the construction of the above cyclotomic cosets from (i) to  
82 (Viii), we can check that these are the only cyclotomic cosets Also, we  
83 can see that order of  $C_0 = \{0\}, (ii) C_{p^n q^m} = \{p^n q^m\}$  is one, i.e.,  $|C_0| = 1$

84  $|C_{p^n q^m}|=1$ , for  $0 \leq k \leq m-1$  order of  $I$  modulo  $q^{m-k} \varphi(q^{m-k})$  so the  
 85 cardinality of  $C_{p^n}$  and  $C_{2p^n}$  is  $\varphi(q^{m-k})$ . Similarly, for  $0 \leq j \leq n-1$ , so the  
 86 cardinality of  $C_{q^m}$  and  $C_{2q^m}$  is  $\varphi(p^{n-j})$ . On Similar lines, we can see  
 87 For  $0 \leq j \leq n-1$ ,  $0 \leq k \leq m-1$  and  $0 \leq w \leq d-1$ ,

$$88 \quad |C_{a^w p^j q^k}| = \frac{\varphi(2p^{n-j} q^{m-k})}{d}, \quad |C_{2a^w p^j q^k}| = \frac{\varphi(2p^{n-j} q^{m-k})}{d},$$

*Then, by order considerations, it follows that the sum*

$$89 \quad |C_0| +$$

$$90 \quad |C_{p^n q^m}| + |C_{2p^n q^m}| + |C_{3p^n q^m}| + |C_{4p^n q^m}| + |C_{5p^n q^m}| + |C_{6p^n q^m}| + |C_{7p^n q^m}|$$

$$91 \quad + |C_{p^n}| + |C_{2p^n}| + |C_{3p^n}| + |C_{4p^n}| + |C_{5p^n}| + |C_{6p^n}| + |C_{7p^n}| +$$

$$92 \quad |C_{q^m}| + |C_{2q^m}| + |C_{3q^m}| + |C_{4q^m}| + |C_{5q^m}| + |C_{6q^m}| + |C_{7q^m}| +$$

$$93 \quad \sum_{j=0}^{n-1} \cdot \sum_{k=0}^{m-1} \cdot \sum_{w=0}^{d-1} \{ |C_{a^w p^j q^k}| +$$

$$94 \quad |C_{2a^w p^j q^k}| + |C_{3a^w p^j q^k}| + |C_{4a^w p^j q^k}| + |C_{5a^w p^j q^k}| + |C_{6a^w p^j q^k}| + |C_{7a^w p^j q^k}| \}.$$

$$95 \quad = 4 + 4 + 8 \sum_{k=0}^{m-1} \cdot \varphi(q^{m-k}) + 8 \sum_{j=0}^{n-1} \cdot \varphi(p^{n-j}) +$$

$$96 \quad 8 \sum_{j=0}^{n-1} \cdot \sum_{k=0}^{m-1} \frac{\varphi(2p^{n-j} q^{m-k})}{d} = 8p^n q^m.$$

## 97 5. REFERENCES

- 98 [1] S.K.Arora, M.Pruthi, "Minimal Cyclic Codes of prime power length",  
 99 Finite Fields Appl.3 (1997)99-113.
- 100 [2] S.K. Arora, M. Pruthi, "Minimal Cyclic Codes of length  $2p^n$ " Finite  
 101 Fields Appl. 5 (1999) 177-187.
- 102 [3] Anuradha Sharma, G.K.Bakshi, V.C. Dumir, M. Raka, "Cyclotomic  
 103 Numbers and Primitive idempotents in the ring  $GF(I)[x]/\langle x^{p^n} - 1 \rangle$ ,  
 104 Finite Fields Appl. 10 (2004) 653-673.
- 105 [4] G.K.Bakshi, Madhu Raka, "Minimal cyclic codes of length  $p^n q$ ", Finite  
 106 Fields Appl. 9 (2003) 432-448.

- 107 [5] A.Sahni and P.T.Sehgal, "Minimal Cyclic Codes of length  $p^nq$ ," *Finite*  
108 *Fields*
- 109 *Appl. 18 (2012) 1017-1036.*
- 110 [6] G.K.Bakshi, Madhu Raka, "Idempotent Generators of Irreducible  
111 Cyclic Codes" *Ramanujan Math Soc, Mysore (2008) 13-18.*
- 112 [7] Ranjeet Singh, M.Pruthi, "Primitive Idempotents of Irreducible  
113 Quadratic Residue Cyclic Codes of Length  $p^nq^m$ " *International Journal of*  
114 *Algebra vol.5 (2011) 285-294.*
- 115 [8] F.J. Mac Williams & N.J.A. Sloane; *The Theory of Error-Correcting*  
116 *Codes, Bell*
- 117 *Laboratories, Murray Hill NJ 07974 U.S.A.*
- 118 [9] Vera Pless, "Introduction to the Theory of Error-Correcting Codes",  
119 *Wiley-*
- 120 *Intersci. Ser. Discrete Math. Optim., (1998).*
- 121 [10] A.Sahni and P.T.Sehgal, "Minimal Cyclic Codes of length  $pnq$ ," *Finite*  
122 *Fields*
- 123 *Appl. 18 (2012) 1017-1036.*
- 124 [11] J. Singh, S. Ahlawat, Some cyclic codes of length  $4p n$  and their  
125 minimum distance bounds, *Int. J. Appl. Eng. Res. 13 (2018), 16708-*  
126 *16718.*
- 127 [12] J. Singh, S.K. Arora, Minimal cyclic codes of length  $8p n$  over  $GF(q)$ ,  
128 where  $q$  is prime power of the form  $8k +5$ , *J. Appl. Math. Comput. 48*  
129 *(2015), 55-69.*

130 [13] J. Singh, S.K. Arora, S. Chawla, Some cyclic codes of length  $8p n$  , Int.  
131 J. Pure Appl. Math. 116 (2017), 217-241.

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