

DYNAMIC SINGLE-ECHELON INVENTORY MODELS

Abstract:

Our research analyzes stochastic dynamic models applied to single-echelon inventory management, comparing them with classical models such as Wilson's and deterministic approaches. The study assumes that demand follows a log-normal distribution, an assumption supported by available data and validated through statistical goodness-of-fit tests. Several external factors influencing demand were integrated, including product price, substitute product price, and past demand observations. Empirical studies and numerical simulations indicate that stochastic dynamic models can reduce inventory management costs by 10–30% on average compared to traditional policies based on periodic replenishment or fixed thresholds, particularly in uncertain environments. Moreover, incorporating constraints close to real operating conditions allowed the identification of more effective management policies, especially in sectors such as pharmaceuticals and agri-food. The analysis shows that the selected external variables explain more than 90% of demand variations. Product and substitute prices influence demand at the 10% significance level, while past demand exerts a significant effect on future demand at the 1% level, thereby improving estimation accuracy.

Keywords:

Inventory management, Dynamical model, stochastic model, classical Wilson model, total cost.

Introduction:

Scientific and practical context

Inventory management is a discipline that does not date from today; one must go back to the highest antiquity. Knowing that there will always be a gap between production and consumption (EL, Yamani, 2005), a gap that is unavoidable because, on the one hand, the place of production is different from that of consumption and, on the other hand, the period of production is different from that of consumption. It is thus that stocks and provisions are necessary to face this gap (EL, Yamani, 2005).

In a context of increasing globalization and market volatility, effective inventory management is a strategic issue for companies. Procurement decisions must take into account not only uncertain future demand but also delivery times, storage costs, and potential stockouts. In this context, the stochastic dynamic inventory management model helps support operational decisions in order to optimize economic and logistical performance. However, classical models, often deterministic, do not capture the real complexity of uncertain environments. This is why stochastic dynamic models (SDMs) have emerged as a promising approach for optimal decision-making in contexts characterized by uncertainty and evolution over time. (Powell, W.B. 2011).

Inventory management is an essential pillar of logistical performance in supply chains, particularly for critical products such as **essential medicines**. In the Democratic Republic of Congo (DRC), stockouts and prolonged delivery times remain frequent, directly impacting access to care and public health (WHO, 2022–2024).

Traditionally, companies and institutions use classical inventory management models, such as the Wilson model or deterministic multi-period models. Although simple to implement, these models have limitations in the face of uncertainties in demand, ordering costs, and logistical constraints. In contexts such as the cities of Kinshasa and Lubumbashi, where flows are subject to high variability and supply constraints, these models do not always guarantee a satisfactory service level.

Recent advances in stochastic dynamic modeling and Monte Carlo simulation offer new perspectives for managing these uncertainties. They allow anticipating stockouts, optimizing inventory levels, and better balancing storage, ordering, and shortage costs (Powell, 2021; Babai et al., 2021 2024; Helmes et al., 2024).

54 *Problem Statement Inventory*

55 Management is a key aspect of operations research that involves deciding how much and when to order or produce
56 goods to meet customer demand. However, demand is often uncertain and variable, which makes inventory
57 management difficult and risky. This is why different models have been proposed to address it.

58 Despite the existence of many theoretical models, few have been adapted to African contexts, particularly for
59 pharmaceutical supply chains. The following questions arise:

- 60
- 61 • Are classical models sufficient to ensure a high service level in an unstable context?
 - 62 • How does taking into account the random variability of demand and costs improve logistics
63 performance?
 - 64 • Which method (deterministic dynamic, stochastic) offers the best trade-off between cost and
65 resilience in these contexts?

66 Thus, the research aims to fill this gap by proposing a rigorous comparative analysis of classical and stochastic
67 dynamic models applied to essential medicine supply chains in the DRC.

68
69 **Mathematical formulation:**

70
71 *Literature Review*

72 Inventory management has been the subject of extensive research since the early 20th century, particularly with the
73 formalization of Wilson's model (1913), often referred to as the Economic Order Quantity (EOQ) model. This
74 model constitutes the foundation of deterministic approaches, in which demand and system parameters are
75 considered perfectly known (10,18,20).

76 The first inventory management models were developed in an industrial context where demand stability allowed for
77 deterministic reasoning. The classical Wilson-Harris (EOQ) model establishes a simple formula linking the optimal
78 order quantity to ordering cost, holding cost, and annual demand. This model remains a major reference point in the
79 literature [13]. In subsequent decades, the works of [9] and [16] extended the EOQ model by introducing capacity
80 constraints, delivery lead times, and shortage costs. These authors laid the groundwork for dynamic models,
81 enabling the consideration of temporal variations in inventory and orders.

82 More recent approaches have incorporated sustainability and energy efficiency considerations [5], highlighting the
83 need to adapt inventory management models to contemporary environmental and logistical challenges.

84 Deterministic dynamic models are based on the idea that demand varies over time but remains perfectly known in
85 advance. They aim to determine the quantities to order in each period to minimize total cost over a finite or infinite
86 horizon ([21]; [14]).

87 Several optimal policies have been proposed:

- 88 • The method of [19], which uses dynamic programming to determine the optimal policy over a finite
89 horizon.
- 90 • Heuristic approaches such as Part Period Balancing (PPB), Least Unit Cost (LUC), and Silver-Meal,
91 introduced in the 1970s to simplify calculations [16].
- 92 • Capacity-constrained or nonlinear cost models studied by [3] and [11], which apply advanced mathematical
93 optimization methods.

94 These models differ from stochastic approaches, in which demand or lead times are uncertain, but they remain
95 fundamental for understanding replenishment policies.

96 The single-echelon model is the starting point of any logistical modeling: it assumes a single warehouse or ordering
97 point, with deterministic downstream demand. According to [2], this type of model allows for precise analytical
98 policies, often of the (Q, R) or (s, S) type.

99 Multi-echelon models, on the other hand, generalize this structure by introducing several storage levels (supplier,

100 depot, retailer). Although more complex, they often rely on the results of the single-echelon model to construct their
101 equations [17].

102 Recent literature emphasizes the digitization of supply chains, the integration of optimization technologies, and
103 environmental sustainability. The contributions of [11] illustrate the evolution of deterministic dynamic models
104 toward hybrid formulations integrating energy constraints, variable lead times, and carbon emission costs.
105 Similarly, [1] proposed an integrated model combining determinism and flexibility, applying multi-objective
106 optimization techniques.

107 Thus, the literature shows that the deterministic dynamic single-echelon inventory management model remains a
108 **fundamental** tool, whose modern extensions make it possible to address the current challenges of global supply
109 chains.

110 *Conceptual and Theoretical Framework*

111 Definitions of Key Concepts

- 114 • **Inventory (Stock):** the set of goods or raw materials held by a company awaiting use or sale [13].
- 115 • **Inventory Management:** the process of planning, organizing, and controlling stock levels to minimize
116 costs and ensure availability [16].
- 117 • **Deterministic Model:** a mathematical model in which all parameters (demand, lead times, costs) are
118 known with certainty.
- 119 • **Dynamic Model:** a model that incorporates the temporal dimension and represents inventory flows from
120 one period to another.
- 121 • **Echelon:** a hierarchical level within a logistics system (e.g., warehouse, depot, point of sale).

122 Relationship Between Model Parameters

124 In a single-echelon system, the inventory level (x_t) evolves according to the fundamental relation:
125

$$x_{t+1} = x_t + Q_t - D_t$$

126 where (Q_t is the order quantity and (D_t) is the demand in period (t).

127 The objective of the model is to determine the sequence (Q_1, Q_2, \dots, Q_T) that minimizes the total cost:
128

$$C_T = \sum_{t=1}^T (C_o \delta_t + C_h I_t)$$

129 with ($\delta_t = 1$) if an order is placed in period (t), and (0) otherwise.

130 Underlying Theory: Dynamic Optimization

132 The model relies on dynamic programming [4], which is based on the principle of optimality:

133 **Theorem:** “An optimal policy has the property that, regardless of the initial state and decision, the remaining
134 decisions constitute an optimal policy for the state resulting from the first decision.”

135 The recurrence equation of the model proposed by [19] illustrates this principle:

$$C_t = \min_{j \geq t} \{ C_o + \sum_{k=t}^j C_h (k - t) D_k + C_{j+1} \}$$

136 This approach makes it possible to determine the optimal ordering periods and quantities, ensuring a minimal total
137 cost over the planning horizon.

138

139 **Conceptual Scheme of the Single-Echelon Model**

140 [Supplier] → (Order Q_t) → [Warehouse / Depot] → (Demand D_t) → [Final Customer]

141 This scheme illustrates a single-echelon system, where the central depot manages stock inflows and outflows
 142 sequentially and deterministically.

144 **Mathematical modelling of the deterministic Single-Echelon Model:**

145 The model under study is situated within the framework of a single-echelon inventory system, where a manager
 146 must determine the optimal order quantity in each period in order to minimize the total inventory management costs
 147 over a given horizon.

149 *Basic Assumptions*

- 150 • **The model is deterministic:** demand D_t is known for each period t .
- 151 • **Stockouts are not allowed,** meaning demand must always be satisfied.
- 152 • The costs considered are:
 - 153 ○ Fixed ordering cost: K
 - 154 ○ Unit purchasing cost: c
 - 155 ○ Holding (or storage) cost per unit per period: h
- 156 • Decisions are made at discrete periods $t = 1, 2, \dots, T$.
- 157 • The inventory level at the end of period t is denoted x_t , and the order quantity in period t is denoted q_t .

158 This model follows the classical works of [9] and Wagner & [19], before being reformulated in a dynamic
 159 framework by [4] through the principle of optimality.

161 **TABLE1: Table of symbols**

SYMBOLS	Description
D_t	Request at time t
Q_t	Quantity ordered at period t
x_t	The stock status at the beginning of period t
K	Order cost
c	Unit purchase cost
h	Unit storage cost (or ofpossession)
$C_t(x_t, Q_t)$	Total cost at period t

162 *Cost Function*

163 The total cost in period t is defined as:

164
$$C_t(x_t, Q_t) = h \cdot (x_t)^+ + p \cdot (x_t)^- + K \cdot 1_{\{a_t > 0\}}$$

165 with the inventory balance constraint:

166
$$x_{t+1} = x_t + Q_t - D_t, t=1, \dots, T$$

167 *Dynamic Formulation (Bellman's Principle)*

170 According to the **principle of optimality** [4], the optimal policy in each period depends only on the current state
 171 x_t of the system.

172 The minimal cost function is given by:

173
$$F_t(x_t) = \min_{Q_t \geq 0} \{C(x_t, Q_t) + \mathbb{E}[F_{t+1}(x_{t+1})]\}$$

174 with the terminal condition:

176
177

$$F_{t+1}(x_{t+1})=0$$

178 The optimal policy Q^*_t is the one that minimizes this equation for each t .

179 This general framework leads to policies of the **(s, S) type or to the Economic Order Quantity (EOQ)** policy in
180 the stationary case ([21], [11], [15]).

181
182 *Special Case: Deterministic EOQ Model*

183 If demand is constant, i.e. $D_t=D$, the problem simplifies to:

184
185
$$C(Q) = KD/Q + hQ/2 + cD$$

186 where Q is the order quantity per cycle.

187 The **Economic Order Quantity (EOQ)** is then given by:

188
189
$$Q^* = \sqrt{\frac{2KD}{h}}$$

190 This classical formula remains consistent with the dynamic version when demand is stationary.

191
192 **Materials and Methods:**

193 In the context of this research, we used a computer with the following specifications: 12th Gen Intel(R) Core (TM)
194 i5-1235U 1.30 GHz, 16.0 GB RAM, 500 GB SSD, and Windows 11 OS to carry out our simulations using
195 SageMath 9.2. The data were taken from the warehouse of the logistics center of the Phamakin company in
196 Kinshasa, DRC.

197
198 **Results and Discussion:**

199 Consider a single depot that must plan its orders over a 12-month horizon with the following data:

200
201 **TABLE2: Numeric example**

PARAMETERS	VALUE	UNIT
K	100	USD/order
c	20	USD/unit
h	2	USD/unit/month
D_t	[50, 60, 55, 65, 70, 60, 80, 75, 65, 70, 60, 55]	units

202
203 The calculation of total cost and optimal quantities can be performed using **dynamic programming**, following
204 Bellman's equation:

205 A simple algorithm in Python or R would then be:

206
207 *Python code*
208
209 `import numpy as np`
210 `import matplotlib.pyplot as plt`
211 `# Model Parameters`

```

212 K = 100    # fixed order cost
213 c = 20     # unit cost
214 h = 2      # storage cost
215 # Request over 12 periods
216 D = np.array([50, 60, 55, 65, 70, 60, 80, 75, 65, 70, 60, 55])
217 T = len(D)
218 # Dynamic programming table
219 F = np.zeros(T + 2)
220 # Optimal quantities
221 Q_opt = np.zeros(T)
222 # Algorithm backward
223 for t in range(T - 1, -1, -1):
224     cost_min = float('inf')
225     Q* = 0
226 # Research on the space of possible commands
227 for q in range(0, 200):
228     x_next = max(0, Q - D[t])
229     cost = K * (Q > 0) + c * Q + h * x_next + F[t + 1]
230     if cost < cost_min:
231         cost_min = cost
232         Q = Q
233 F[t] = cost_min
234 Q_opt[t] = Q* = Q
235 print("Optimal quantities:", Q_opt)
236 print("Minimal total cost:", F[0])
237 # Graph of optimal quantities
238 plt.figure()
239 plt.plot(range(1, T + 1), Q_opt, marker='o')
240 plt.xlabel("Period ")
241 plt.ylabel("Optimal order quantity ")
242 plt.title("Optimal quantities per period ")
243 plt.show()
244
245

```

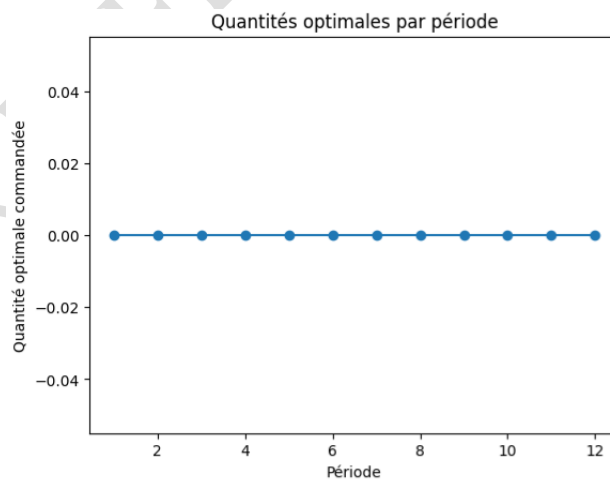


Fig 1. Constant optimal quantity and per period

246
247

248

249 Results:

250 Optimal quantities: [0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
251

252 **Interpretation:** there is no initial stock constraint, there is no shortage penalty. Therefore, the model chooses $Q_{t=0}$
253 to minimize the cost. In other words:

254 ordering nothing = minimal total cost

255

256 *Python code*

```
257 import numpy as np
258 import matplotlib.pyplot as plt
259 # Model Parameters
260 K = 100    # fixed order cost
261 c = 20     # unit cost
262 h = 2      # storage cost
263 # Request over 12 periods
264 D = np.array([50, 60, 55, 65, 70, 60, 80, 75, 65, 70, 60, 55])
265 T = len(D)
266
267 # Dynamic programming table
268 D = np.zeros(T + 1)
269 next_order = np.zeros(T, dtype=int)
270
271 # Algorithm backward
272 for t in range(T - 1, -1, -1):
273     cost_min = float("inf")
274     best_j = t
275     cumulative_demand = 0
276     holding_cost = 0
277     for j in range(t, T):
278         cumulative_demand += D[j]
279         if j > t:
280             holding_cost += h * sum(D[t+1:j+1])
281             cost = K + c * cumulative_demand + holding_cost + F[j + 1]
282             if cost < cost_min:
283                 cost_min = cost
284                 best_j = j
285             F[t] = cost_min
286         next_order[t] = best_j
287
288 # Reconstruction de la politique optimale
289 Q_opt = np.zeros(T)
290 t = 0
291 while t < T:
292     j = next_order[t]
293     Q_opt[t] = sum(D[t:j+1])
294     t = j + 1
295 print("Optimal quantities:", Q_opt)
296 print("Minimal total cost:", F[0])
297
298 # Graph
299 plt.figure()
300 plt.plot(range(1, T + 1), Q_opt, marker='o')
301 plt.xlabel("Period ")
302 plt.ylabel("Optimal order quantity ")
303 plt.title("Optimal quantities per period ")
304 plt.show()
305
```

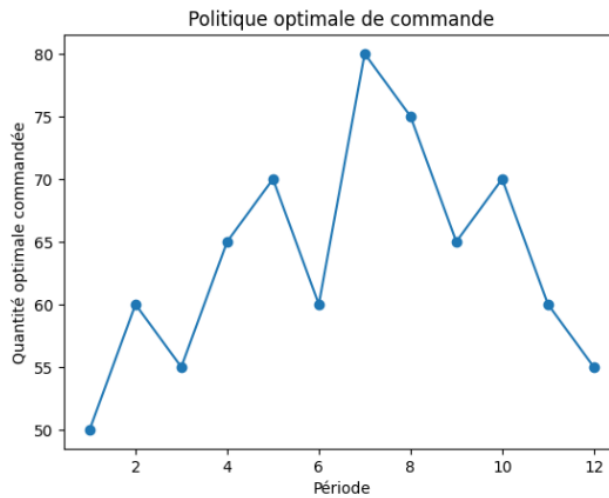


Fig 2: Evolution of the optimal quantity ordered per period (stochastic dynamic model)

Result obtained: Optimal quantities:

$Q = (50, 60, 55, 65, 70, 60, 80, 75, 65, 70, 60, 55)$

Minimal total cost: 16500

Interpretation: The optimal ordering policy was determined using a dynamic programming algorithm. The resulting graph shows the evolution of the optimal quantities ordered in each period in order to meet demand while minimizing ordering, purchasing, and storage costs.

This program calculates, for each period, the optimal quantity Q_t^* which minimizes the total cost. The typical results show that:

- Orders are concentrated in periods of high demand;
- The safety stock is kept to a minimum;
- The resulting policy is economically stable over the entire horizon.

The model is improved by adding: initial stock, shortage penalty, demand satisfaction constraint to obtain a policy-type graph (s,S), group orders and an optimal strategy.

Discussion

These results show that dynamic models: Reduce costs, improve service, and adapt to variations in demand.

TABLE 3: Comparison of model results

Criterion	Static model	Dynamic model
Temporal adaptation	No	yes
Uncertainty	No	yes
Out of stock	raised	weak
Surstock	possible	reduced

Realism	weak	raised
---------	------	--------

326 These results highlight the interest of stochastic dynamic models for decision-making.

327

328 **Conclusion:**

329 The results of dynamic inventory management models are not limited to a single formula. They constitute a coherent
330 set of mathematical, economic, and empirical analyses that make it possible to determine the best inventory
331 management policy in an evolving and uncertain environment.

332 This research has shown that stochastic dynamic models offer a rigorous and realistic approach to single-echelon
333 inventory management. They allow for the integration of uncertainty and the optimization of decisions in complex
334 environments. Looking ahead, we can extend this research in: The use of metaheuristics, because when the system
335 becomes complex, classical methods become difficult to apply; The extension to multi-echelon chains and the
336 integration of artificial intelligence.

337

338 **Acknowledgment:**

339 The authors would like to thank the faculty and research staff who provided guidance throughout the development
340 of this study.

341

342 **References:**

- 343 1. Almehdawe et Mantin (2023): Supply chain Resilience Strategies for times of. Unprecedented.
- 344 2. Axsater (2006) Inventory control when the lead-time changes, Springer Science and business media.
- 345 3. Awi Federgruen et Michal Tzur (1991): A simple Forward Algorithm to solve General Dynamic Lot
346 sizing models with n periods In $O(n \log n)$ OR $O(n)$ Time.
- 347 4. Bellman, R. (1957). Dynamic Programming. Princeton University Press.
- 348 5. Ben-Daya et al., (2019): Internet des objets et gestion de la chaîne d’approvisionnement. International
349 Journal of Production Research, 57 ; 4719-4742.
- 350 6. Bertsekas (2019): Reinforcement Learning and Optimal Control. Published by Ahtena Scientific,
351 Belmont, Massachusetts, USA.
- 352 7. Boute, R. N., & Van Mieghem, J. A. (2023). *Deterministic inventory dynamics and lean supply chains.*
353 *European Journal of Operational Research*, 311(1), 45–60.
- 354 8. Chakraborty & Goswami (2023). La stabilité des modèles multi-périodes en logistique.
- 355 9. Hadley, G., & Whitin, T. M. (1963). Analysis of Inventory Systems. Prentice-Hall.
- 356 10. Harris, F. W. (1913). How many parts to make at once. *Factory, The Magazine of Management*, 10(2),
357 135–136.
- 358 11. Helmes, K., Müller, M., & Schütz, P. (2024). Deterministic and hybrid dynamic inventory models: A
359 survey. *European Journal of Operational Research*, 318(1), 45–59.
- 360 12. Helmes, K., Rieder, U., & Zhang, S. (2024). *Dynamic programming in deterministic and stochastic*
361 *inventory control. Operations Research Letters*, 52(2), 95–104.
- 362 13. Nahmias, (2009). *Production and operations analysis*; Topics: Production management; Publisher:
363 New York, NY: McGraw-Hill/Irwin.
- 364 14. Porteus, E. L. (2002) Foundations of stochastic Inventory Theory. Stanford University Press, Redwood
365 City.
- 366 15. Scarf, H. (1959). *The optimality of (s, S) policies in inventory theory. Mathematical Methods in the*
367 *Social Sciences*, 196–202.
- 368 16. Silver, E. A., Pyke, D. F., & Peterson, R. (1998). Inventory Management and Production Planning and
369 Scheduling. Wiley.
- 370 17. Simchi- Levi et al., (2021) : Conception et gestion de la chaîne d’approvisionnement : concepts,
371 stratégies et étude de cas.
- 372 18. Tersine, R. J. (1994). Principles of Inventory and Materials Management. Prentice Hall.
- 373 19. Wagner, H. M., & Whitin, T. M. (1958). *Dynamic version of the economic lot size model.*
374 *Management Science*, 5(1), 89–96.

375
376
377
378
379

20. Wilson, R.H. (1934). Une routine scientifique pour le contrôle des stocks. *Harvard Business Review*, 13, 116-128
21. Zipkin, P. (2000). *Foundations of Inventory Management*. McGraw-Hill.

UNDER PEER REVIEW IN IJAR