

A Second Derivative Hybrid Block Method with Variational Stability Parameters for the Direct Solution of Second-Order Initial Value Problems in Ordinary Differential Equations

ABSTRACT

This paper presents the derivation and implementation of a second-derivative hybrid block method with stability parameters for the direct solution of second order initial value problems in ordinary differential equations. The method is developed using interpolation and collocation techniques to obtain continuous formulation, which is evaluated at selected grid and off-grid points to generate the discrete block scheme. The convergence and stability properties of the method are investigated, and the results show that the method possesses $A(\alpha)$ -stability, A -stability, and additional regions of absolute stability for different values of the stability parameters. The method is applied to some selected initial value problems, and the results overlap with the exact solution, as illustrated in Figures.4, and.5 respectively. These results demonstrate the accuracy and efficiency of the proposed method when applied to both linear and nonlinear problems.

KEYWORDS: Block method, Interpolation, Collocation, Hybrid and Polynomial basis function.

I. INTRODUCTION

This paper considers Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs) of the form:

$$y'' = f(x, y, y'); \quad y'(x_0) = y'_0, \quad y(x_0) = y_0, \quad x \in [x_0, x_n] \quad (1.1)$$

Equation (1.1) is called second order initial value problem in ordinary differential equation. Finding the direct numerical solution to equation of this form is an interesting area for researchers in literature. Traditionally equation (1.1) together with its associated conditions can be solved by reducing it to its first order equivalent (Dahlquist 1956). The reduced form is then solved using any appropriate numerical method. (Awoyemi 1999), this approach does not fully utilize all the information connected with certain ordinary differential equations, such as oscillatory nature of solution. (Jain 2012) and (Sair 2014) reported that this approach enlarges the size of equations and also causes computational burden in writing computer programmes which may jeopardize the performance of the method in terms of time and error. Hence (Areo and Adeniyi 2014), (Okuonghae and Ikhile 2014), (Bolaji 2017), (Adeyeye and Omar 2017), (Adeyefa & Olagunju 2021), (Kamarun et al 2023), (Taiwo, et al 2023), (Abdulazeez 2024), (Workineh 2024), (Tadema 2024) etc. worked on different methods that solved equation (1.1) but considered fixed stability parameters. However, this article considers the derivation of hybrid block method with variational stability parameters which will help in improving the stability of the method.

2. DERIVATION OF THE METHOD

Consider the polynomial function as:

$$p(x) = \sum_{j=0}^n a_j x^j \quad (2.1)$$

and use it to deduce an approximate solution to (1.1) as:

$$y(x) \cong p(x) = \sum_{j=0}^{q+t-1} a_j x^j \quad (2.2)$$

74 where $a_j, j = 0, 1, \dots$ are unknown parameters to be determined, $n = q + t - 1$, q and t are the points
75 of interpolation and collocation, respectively. Then, the second derivative of (2.2):

$$y''(x) = f(x, y, y') = \sum_{j=1}^{q+t-1} j(j-1)a_j x_{n+i}^{j-2} i = \left(0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right) \quad (2.3)$$

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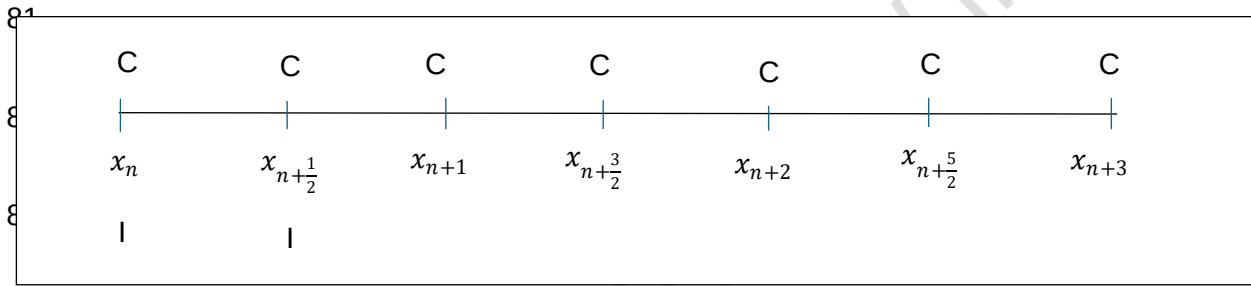
$$y_{n+i} = \sum_{j=0}^{q+t-1} a_j x_{n+i}^j i = \left(0, \frac{1}{2}\right) \quad (2.4)$$

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$$f_{n+i} = \sum_{j=1}^{q+t-1} j a_j x_{n+i}^{j-1} i = \left(0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right) \quad (2.5)$$

79 The interpolation of equation (2.4) at the point $x = x_n, x_{n+\frac{1}{2}}$ and collocation of equation (2.5) at

80 $x = \left(0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right)$ for $q = 2$ and $t = 7$ i.e.



84 will give a system of equations:

$$a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + a_6 x_n^6 + a_7 x_n^7 + a_8 x_n^8 = y_n$$

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$$\begin{aligned} a_0 + \frac{ha_1}{2} + \frac{h^2 a_2}{4} + \frac{h^3 a_3}{8} + \frac{h^4 a_4}{16} + \frac{h^5 a_5}{32} + \frac{h^6 a_6}{64} + \frac{h^7 a_7}{128} + \frac{h^8 a_8}{256} + a_1 x_n + ha_2 x_n + \frac{3}{4} h^2 a_3 x_n + \frac{1}{2} h^3 a_4 x_n \\ + \frac{5}{16} h^4 a_5 x_n + \frac{3}{16} h^5 a_6 x_n + \frac{7}{64} h^6 a_7 x_n + \frac{1}{16} h^7 a_8 x_n + a_2 x_n^2 + \frac{3}{2} ha_3 x_n^2 + \frac{3}{2} h^2 a_4 x_n^2 \\ + \frac{5}{4} h^3 a_5 x_n^2 + \frac{15}{16} h^4 a_6 x_n^2 + \frac{21}{32} h^5 a_7 x_n^2 + \frac{7}{16} h^6 a_8 x_n^2 + a_3 x_n^3 + 2ha_4 x_n^3 + \frac{5}{2} h^2 a_5 x_n^3 \\ + \frac{5}{2} h^3 a_6 x_n^3 + \frac{35}{16} h^4 a_7 x_n^3 + \frac{7}{4} h^5 a_8 x_n^3 + a_4 x_n^4 + \frac{5}{2} ha_5 x_n^4 + \frac{15}{4} h^2 a_6 x_n^4 + \frac{35}{8} h^3 a_7 x_n^4 \\ + \frac{35}{8} h^4 a_8 x_n^4 + a_5 x_n^5 + 3ha_6 x_n^5 + \frac{21}{4} h^2 a_7 x_n^5 + 7h^3 a_8 x_n^5 + a_6 x_n^6 + \frac{7}{2} ha_7 x_n^6 + 7h^2 a_8 x_n^6 \\ + a_7 x_n^7 + 4ha_8 x_n^7 + a_8 x_n^8 = y_{n+\frac{1}{2}} \end{aligned}$$

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$$\begin{aligned} 2a_2 + 6a_3 x_n + 12a_4 x_n^2 + 20a_5 x_n^3 + 30a_6 x_n^4 + 42a_7 x_n^5 + 56a_8 x_n^6 = f_n \\ 2a_2 + 3ha_3 + 3h^2 a_4 + \frac{5h^3 a_5}{2} + \frac{15h^4 a_6}{8} + \frac{21h^5 a_7}{16} + \frac{7h^6 a_8}{8} + 6a_3 x_n + 12ha_4 x_n + 15h^2 a_5 x_n + 15h^3 a_6 x_n \\ + \frac{105}{8} h^4 a_7 x_n + \frac{21}{2} h^5 a_8 x_n + 12a_4 x_n^2 + 30ha_5 x_n^2 + 45h^2 a_6 x_n^2 + \frac{105}{2} h^3 a_7 x_n^2 \\ + \frac{105}{2} h^4 a_8 x_n^2 + 20a_5 x_n^3 + 60ha_6 x_n^3 + 105h^2 a_7 x_n^3 + 140h^3 a_8 x_n^3 + 30a_6 x_n^4 + 105ha_7 x_n^4 \\ + 210h^2 a_8 x_n^4 + 42a_7 x_n^5 + 168ha_8 x_n^5 + 56a_8 x_n^6 = f_{n+\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& 2a_2 + 6ha_3 + 12h^2a_4 + 20h^3a_5 + 30h^4a_6 + 42h^5a_7 + 56h^6a_8 + 6a_3x_n + 24ha_4x_n + 60h^2a_5x_n \\
& + 120h^3a_6x_n + 210h^4a_7x_n + 336h^5a_8x_n + 12a_4x_n^2 + 60ha_5x_n^2 + 180h^2a_6x_n^2 \\
& + 420h^3a_7x_n^2 + 840h^4a_8x_n^2 + 20a_5x_n^3 + 120ha_6x_n^3 + 420h^2a_7x_n^3 + 1120h^3a_8x_n^3 \\
& + 30a_6x_n^4 + 210ha_7x_n^4 + 840h^2a_8x_n^4 + 42a_7x_n^5 + 336ha_8x_n^5 + 56a_8x_n^6 = f_{n+1}
\end{aligned}$$

$$\begin{aligned}
& 2a_2 + 9ha_3 + 27h^2a_4 + \frac{135h^3a_5}{2} + \frac{1215h^4a_6}{8} + \frac{5103h^5a_7}{16} + \frac{5103h^6a_8}{8} + 6a_3x_n + 36ha_4x_n + 135h^2a_5x_n \\
& + 405h^3a_6x_n + \frac{8505}{8}h^4a_7x_n + \frac{5103}{2}h^5a_8x_n + 12a_4x_n^2 + 90ha_5x_n^2 + 405h^2a_6x_n^2 \\
& + \frac{2835}{2}h^3a_7x_n^2 + \frac{8505}{2}h^4a_8x_n^2 + 20a_5x_n^3 + 180ha_6x_n^3 + 945h^2a_7x_n^3 + 3780h^3a_8x_n^3 \\
& + 30a_6x_n^4 + 315ha_7x_n^4 + 1890h^2a_8x_n^4 + 42a_7x_n^5 + 504ha_8x_n^5 + 56a_8x_n^6 = f_{n+\frac{3}{2}}
\end{aligned}$$

$$\begin{aligned}
& 2a_2 + 12ha_3 + 48h^2a_4 + 160h^3a_5 + 480h^4a_6 + 1344h^5a_7 + 3584h^6a_8 + 6a_3x_n + 48ha_4x_n + 240h^2a_5x_n \\
& + 960h^3a_6x_n + 3360h^4a_7x_n + 10752h^5a_8x_n + 12a_4x_n^2 + 120ha_5x_n^2 + 720h^2a_6x_n^2 \\
& + 3360h^3a_7x_n^2 + 13440h^4a_8x_n^2 + 20a_5x_n^3 + 240ha_6x_n^3 + 1680h^2a_7x_n^3 + 8960h^3a_8x_n^3 \\
& + 30a_6x_n^4 + 420ha_7x_n^4 + 3360h^2a_8x_n^4 + 42a_7x_n^5 + 672ha_8x_n^5 + 56a_8x_n^6 = f_{n+2}
\end{aligned}$$

$$\begin{aligned}
& 2a_2 + 15ha_3 + 75h^2a_4 + \frac{625h^3a_5}{2} + \frac{9375h^4a_6}{8} + \frac{65625h^5a_7}{16} + \frac{109375h^6a_8}{8} + 6a_3x_n + 60ha_4x_n \\
& + 375h^2a_5x_n + 1875h^3a_6x_n + \frac{65625}{8}h^4a_7x_n + \frac{65625}{2}h^5a_8x_n + 12a_4x_n^2 + 150ha_5x_n^2 \\
& + 1125h^2a_6x_n^2 + \frac{13125}{2}h^3a_7x_n^2 + \frac{65625}{2}h^4a_8x_n^2 + 20a_5x_n^3 + 300ha_6x_n^3 + 2625h^2a_7x_n^3 \\
& + 17500h^3a_8x_n^3 + 30a_6x_n^4 + 525ha_7x_n^4 + 5250h^2a_8x_n^4 + 42a_7x_n^5 + 840ha_8x_n^5 \\
& + 56a_8x_n^6 = f_{n+\frac{5}{2}}
\end{aligned}$$

$$\begin{aligned}
& 2a_2 + 18ha_3 + 108h^2a_4 + 540h^3a_5 + 2430h^4a_6 + 10206h^5a_7 + 40824h^6a_8 + 6a_3x_n + 72ha_4x_n \\
& + 540h^2a_5x_n + 3240h^3a_6x_n + 17010h^4a_7x_n + 81648h^5a_8x_n + 12a_4x_n^2 + 180ha_5x_n^2 \\
& + 1620h^2a_6x_n^2 + 11340h^3a_7x_n^2 + 68040h^4a_8x_n^2 + 20a_5x_n^3 + 360ha_6x_n^3 + 3780h^2a_7x_n^3 \\
& + 30240h^3a_8x_n^3 + 30a_6x_n^4 + 630ha_7x_n^4 + 7560h^2a_8x_n^4 + 42a_7x_n^5 + 1008ha_8x_n^5 \\
& + 56a_8x_n^6 = f_{n+3}
\end{aligned}$$

87 Now like terms of the above expansion are collected and presented in matrix form as:

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\
1 & \left(x_n + \frac{h}{2}\right) & \left(x_n + \frac{h}{2}\right)^2 & \left(x_n + \frac{h}{2}\right)^3 & \left(x_n + \frac{h}{2}\right)^4 & \left(x_n + \frac{h}{2}\right)^5 & \left(x_n + \frac{h}{2}\right)^6 & \left(x_n + \frac{h}{2}\right)^7 & \left(x_n + \frac{h}{2}\right)^8 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\
0 & 0 & 2 & 6\left(x_n + \frac{h}{2}\right) & 12\left(x_n + \frac{h}{2}\right)^2 & 20\left(x_n + \frac{h}{2}\right)^3 & 30\left(x_n + \frac{h}{2}\right)^4 & 42\left(x_n + \frac{h}{2}\right)^5 & 56\left(x_n + \frac{h}{2}\right)^6 \\
0 & 1 & 2 & 6\left(x_n + h\right) & 12\left(x_n + h\right)^2 & 20\left(x_n + h\right)^3 & 30\left(x_n + h\right)^4 & 42\left(x_n + h\right)^5 & 56\left(x_n + h\right)^6 \\
0 & 1 & 2 & 6\left(x_n + \frac{3h}{2}\right) & 12\left(x_n + \frac{3h}{2}\right)^2 & 20\left(x_n + \frac{3h}{2}\right)^3 & 30\left(x_n + \frac{3h}{2}\right)^4 & 42\left(x_n + \frac{3h}{2}\right)^5 & 56\left(x_n + \frac{3h}{2}\right)^6 \\
0 & 1 & 2 & 6\left(x_n + 2h\right) & 12\left(x_n + 2h\right)^2 & 20\left(x_n + 2h\right)^3 & 30\left(x_n + 2h\right)^4 & 42\left(x_n + 2h\right)^5 & 56\left(x_n + 2h\right)^6 \\
0 & 1 & 2 & 6\left(x_n + \frac{5h}{2}\right) & 12\left(x_n + \frac{5h}{2}\right)^2 & 20\left(x_n + \frac{5h}{2}\right)^3 & 30\left(x_n + \frac{5h}{2}\right)^4 & 42\left(x_n + \frac{5h}{2}\right)^5 & 56\left(x_n + \frac{5h}{2}\right)^6 \\
0 & 1 & 2 & 6\left(x_n + 3h\right) & 12\left(x_n + 3h\right)^2 & 20\left(x_n + 3h\right)^3 & 30\left(x_n + 3h\right)^4 & 42\left(x_n + 3h\right)^5 & 56\left(x_n + 3h\right)^6
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7 \\
a_8
\end{bmatrix}$$

$$= \begin{bmatrix}
y_n \\
y_{n+\frac{1}{2}} \\
f_n \\
f_{n+\frac{1}{2}} \\
f_{n+1} \\
f_{n+\frac{3}{2}} \\
f_{n+2} \\
f_{n+\frac{5}{2}} \\
f_{n+3}
\end{bmatrix} \quad (2.6)$$

88 Solving (2.6) using Gaussian elimination method yields:

$$y(x) = \alpha_0 y_n + \alpha_1 y_{n+\frac{1}{2}} + h^2 \left[\beta_0 f_n + \beta_1 f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_3 f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_5 f_{n+\frac{5}{2}} + \beta_3 f_{n+3} \right] \quad (2.7)$$

89 Where:

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$$\begin{aligned}
& \alpha_0 = 1 - 2t \\
& \alpha_{\frac{1}{2}} = 2t \\
& \beta_0 = \frac{h^2 t(-1+2t) \left(28549 + 2t \left(-31931 + 2t \left(17461 + 2t \left(-5275 + 2t(899 + 6t(-27+2t)) \right) \right) \right) \right) \right)}{241920} \\
& \beta_{\frac{1}{2}} = - \frac{h^2 t \left(9625 + 64(-4+t)t^2 \left(315 + 2t(-189 + t(105 + t(-28+3t))) \right) \right)}{40320} \\
& \beta_1 = \frac{h^2 t \left(17151 + 32t^2 \left(-6300 + t \left(12285 + t(-9681 + 4t(959 + 5t(-38+3t))) \right) \right) \right)}{80640} \\
& \beta_{\frac{3}{2}} = - \frac{h^2 t \left(10621 + 64t^2 \left(-2100 + t \left(4445 + 2t(-1953 + t(847 + 15(-12+t)t)) \right) \right) \right)}{60480} \\
& \beta_2 = \frac{h^2 t \left(7703 + 32t^2 \left(-3150 + t \left(6930 + t(-6447 + 4t(749 + 5t(-34+3t))) \right) \right) \right)}{80640} \\
& \beta_{\frac{5}{2}} = - \frac{h^2 t \left(1209 + 64t^2 \left(-252 + t \left(567 + 2t(-273 + t(133 + t(-32+3t))) \right) \right) \right)}{40320} \\
& \beta_3 = \frac{h^2 t(995 + 32(-3+t)t^2(140 + t(-273 + 4t(56 + 3(-7+t)t))))}{241920}
\end{aligned} \tag{2.8}$$

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Evaluating $\alpha_0(t)$, $\alpha_{\frac{1}{2}}(t)$, $\beta_0(t)$, $\beta_{\frac{1}{2}}(t)$, $\beta_1(t)$, $\beta_{\frac{3}{2}}(t)$, $\beta_2(t)$, $\beta_{\frac{5}{2}}(t)$,

$\beta_3(t)$ at the non – interpolating points i.e. at x_{n+1} , $x_{n+\frac{3}{2}}$, x_{n+2} , $x_{n+\frac{5}{2}}$, x_{n+3}

93 Using the transformation:

$$94 \quad x = ht + x_n \tag{2.9}$$

95 Now replace x in equation (2.9) with each of the collocating points to obtain the corresponding
96 values of t

97 For x_{n+1}

$$x_n + h = ht + x_n \Rightarrow h = ht \Rightarrow t = 1$$

98 For $x_{n+\frac{3}{2}}$

$$x_n + \frac{3h}{2} = ht + x_n \Rightarrow \frac{3h}{2} = ht \Rightarrow t = \frac{3}{2}$$

99 For x_{n+2}

$$x_n + 2h = ht + x_n \Rightarrow 2h = ht \Rightarrow t = 2$$

100 For $x_{n+\frac{5}{2}}$

$$x_n + \frac{5h}{2} = ht + x_n \Rightarrow \frac{5h}{2} = ht \Rightarrow t = \frac{5}{2}$$

101 For x_{n+3}

$$x_n + 3h = ht + x_n \Rightarrow 3h = ht \Rightarrow t = 3$$

102 i.e. $t = 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ produces the following schemes:

$$\begin{aligned}
y_{n+1} &= -y_n + 2y_{\frac{1}{2}+n} + \frac{863h^2f_n}{48384} + \frac{8999h^2f_{\frac{1}{2}+n}}{40320} - \frac{769h^2f_{1+n}}{80640} + \frac{1987h^2f_{\frac{3}{2}+n}}{60480} - \frac{1609h^2f_{2+n}}{80640} + \frac{263h^2f_{\frac{5}{2}+n}}{40320} - \frac{221h^2f_{3+n}}{241920} \\
y_{n+\frac{3}{2}} &= -2y_n + 3y_{\frac{1}{2}+n} + \frac{2803h^2f_n}{80640} + \frac{1265h^2f_{\frac{1}{2}+n}}{2688} + \frac{1657h^2f_{1+n}}{8960} + \frac{1777h^2f_{\frac{3}{2}+n}}{20160} - \frac{1049h^2f_{2+n}}{26880} + \frac{11}{896}h^2f_{\frac{5}{2}+n} - \frac{137h^2f_{3+n}}{80640} \\
y_{n+2} &= -3y_n + 4y_{\frac{1}{2}+n} + \frac{2089h^2f_n}{40320} + \frac{4813h^2f_{\frac{1}{2}+n}}{6720} + \frac{5461h^2f_{1+n}}{13440} + \frac{3457h^2f_{\frac{3}{2}+n}}{10080} - \frac{419h^2f_{2+n}}{13440} + \frac{109h^2f_{\frac{5}{2}+n}}{6720} - \frac{19h^2f_{3+n}}{8064} \\
y_{n+\frac{5}{2}} &= -4y_n + 5y_{\frac{1}{2}+n} + \frac{1669h^2f_n}{24192} + \frac{3875h^2f_{\frac{1}{2}+n}}{4032} + \frac{5069h^2f_{1+n}}{8064} + \frac{3751h^2f_{\frac{3}{2}+n}}{6048} + \frac{1457h^2f_{2+n}}{8064} + \frac{179h^2f_{\frac{5}{2}+n}}{4032} - \frac{95h^2f_{3+n}}{24192} \\
y_{n+3} &= -5y_n + 6y_{\frac{1}{2}+n} + \frac{1375h^2f_n}{16128} + \frac{3259h^2f_{\frac{1}{2}+n}}{2688} + \frac{1489h^2f_{1+n}}{1792} + \frac{3751h^2f_{\frac{3}{2}+n}}{4032} + \frac{2059h^2f_{2+n}}{5376} + \frac{265}{896}h^2f_{\frac{5}{2}+n} + \frac{199h^2f_{3+n}}{16128}
\end{aligned}$$

103 Differentiating (2.8)

104

105

$$\begin{aligned}
&\alpha'_0 = -2 \\
&\alpha'_1 = 2 \\
&\beta'_0 = -\frac{28549h^2}{241920} + h^2t - \frac{49h^2t^2}{20} + \frac{406h^2t^3}{135} - \frac{49h^2t^4}{24} + \frac{7h^2t^5}{9} - \frac{7h^2t^6}{45} + \frac{4h^2t^7}{315} \\
&\beta'_{\frac{1}{2}} = -\frac{275h^2}{1152} + 6h^2t^2 - \frac{58h^2t^3}{5} + \frac{29h^2t^4}{3} - \frac{62h^2t^5}{15} + \frac{8h^2t^6}{9} - \frac{8h^2t^7}{105} \\
&\beta'_1 = \frac{5717h^2}{26880} - \frac{15h^2t^2}{2} + \frac{39h^2t^3}{2} - \frac{461h^2t^4}{24} + \frac{137h^2t^5}{15} - \frac{19h^2t^6}{9} + \frac{4h^2t^7}{21} \\
&\beta'_{\frac{3}{2}} = -\frac{10621h^2}{60480} + \frac{20h^2t^2}{3} - \frac{508h^2t^3}{27} + \frac{62h^2t^4}{3} - \frac{484h^2t^5}{45} + \frac{8h^2t^6}{3} - \frac{16h^2t^7}{63} \\
&\beta'_2 = \frac{7703h^2}{80640} - \frac{15h^2t^2}{4} + 11h^2t^3 - \frac{307h^2t^4}{24} + \frac{107h^2t^5}{15} - \frac{17h^2t^6}{9} + \frac{4h^2t^7}{21} \\
&\beta'_{\frac{5}{2}} = -\frac{403h^2}{13440} + \frac{6h^2t^2}{5} - \frac{18h^2t^3}{5} + \frac{13h^2t^4}{3} - \frac{38h^2t^5}{15} + \frac{32h^2t^6}{45} - \frac{8h^2t^7}{105} \\
&\beta'_3 = \frac{199h^2}{48384} - \frac{h^2t^2}{6} + \frac{137h^2t^3}{270} - \frac{5h^2t^4}{8} + \frac{17h^2t^5}{45} - \frac{h^2t^6}{9} + \frac{4h^2t^7}{315}
\end{aligned} \quad (2.11)$$

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107 Equation (2.11) is evaluated at all the grid points i.e. at $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ using the derivative of (2.7)

108 i.e.

$$hy'_n = \alpha'_0 y_n + \alpha'_1 y_{n+\frac{1}{2}} + h^2 \left[\beta'_0 f_n + \beta'_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta'_1 f_{n+1} + \beta'_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta'_2 f_{n+2} + \beta'_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta'_3 f_{n+3} \right] \quad (2.12)$$

109 which gives:

$$\begin{aligned}
hy'_n &= -2y_n + 2y_{n+\frac{1}{2}} + h^2 \left[-\frac{28549}{241920}f_n - \frac{275}{1152}f_{n+\frac{1}{2}} + \frac{5717}{26880}f_{n+1} - \frac{10621}{60480}f_{n+\frac{3}{2}} + \frac{7703}{80640}f_{n+2} - \frac{403}{13440}f_{n+\frac{5}{2}} + \frac{199}{48384}f_{n+3} \right] \\
hy'_{n+\frac{1}{2}} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2 \left[\frac{275}{6912}f_n + \frac{12079}{40320}f_{n+\frac{1}{2}} - \frac{13823}{80640}f_{n+1} + \frac{8131}{60480}f_{n+\frac{3}{2}} - \frac{5771}{80640}f_{n+2} + \frac{179}{8064}f_{n+\frac{5}{2}} - \frac{731}{241920}f_{n+3} \right] \\
hy'_{n+1} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2 \left[\frac{2633}{80640}f_n + \frac{4091}{8064}f_{n+\frac{1}{2}} + \frac{17503}{80640}f_{n+1} + \frac{1}{20160}f_{n+\frac{3}{2}} - \frac{181}{16128}f_{n+2} + \frac{199}{40320}f_{n+\frac{5}{2}} - \frac{1}{1280}f_{n+3} \right] \\
hy'_{n+\frac{3}{2}} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2 \left[\frac{8441}{241920}f_n + \frac{3907}{8064}f_{n+\frac{1}{2}} + \frac{12683}{26880}f_{n+1} + \frac{3751}{12096}f_{n+\frac{3}{2}} - \frac{5419}{80640}f_{n+2} + \frac{7}{384}f_{n+\frac{5}{2}} - \frac{571}{241920}f_{n+3} \right] \\
hy'_{n+2} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2 \left[\frac{8059}{241920}f_n + \frac{20071}{40320}f_{n+\frac{1}{2}} + \frac{6707}{16128}f_{n+1} + \frac{37507}{60480}f_{n+\frac{3}{2}} + \frac{2161}{11520}f_{n+2} - \frac{37}{8064}f_{n+\frac{5}{2}} - \frac{29}{241920}f_{n+3} \right] \\
hy'_{n+\frac{5}{2}} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2 \left[\frac{2867}{80640}f_n + \frac{3875}{8064}f_{n+\frac{1}{2}} + \frac{38401}{80640}f_{n+1} + \frac{1399}{2880}f_{n+\frac{3}{2}} + \frac{46453}{80640}f_{n+2} + \frac{8191}{40320}f_{n+\frac{5}{2}} - \frac{13}{1792}f_{n+3} \right] \\
hy'_{n+3} &= -2y_n + 2y_{n+\frac{1}{2}} + h^2 \left[\frac{1375}{48384}f_n + \frac{21479}{40320}f_{n+\frac{1}{2}} + \frac{1187}{3840}f_{n+1} + \frac{48131}{60480}f_{n+\frac{3}{2}} + \frac{15479}{80640}f_{n+2} + \frac{1993}{2688}f_{n+\frac{5}{2}} + \frac{36419}{241920}f_{n+3} \right]
\end{aligned} \quad (2.13)$$

110 Now combining Equation (2.10) and (2.13) gives:

$$\begin{aligned}
y_{n+1} - 2y_{\frac{1}{2}+n} &= -y_n + \frac{863h^2f_n}{48384} + \frac{8999h^2f_{\frac{1}{2}+n}}{40320} - \frac{769h^2f_{1+n}}{80640} + \frac{1987h^2f_{\frac{3}{2}+n}}{60480} - \frac{1609h^2f_{2+n}}{80640} + \frac{263h^2f_{\frac{5}{2}+n}}{40320} - \frac{221h^2f_{3+n}}{241920} \\
y_{n+\frac{3}{2}} - 3y_{\frac{1}{2}+n} &= -2y_n + \frac{2803h^2f_n}{80640} + \frac{1265h^2f_{\frac{1}{2}+n}}{2688} + \frac{1657h^2f_{1+n}}{8960} + \frac{1777h^2f_{\frac{3}{2}+n}}{20160} - \frac{1049h^2f_{2+n}}{26880} + \frac{11}{896}h^2f_{\frac{5}{2}+n} - \frac{137h^2f_{3+n}}{80640} \\
y_{n+2} - 4y_{\frac{1}{2}+n} &= -3y_n + \frac{2089h^2f_n}{40320} + \frac{4813h^2f_{\frac{1}{2}+n}}{6720} + \frac{5461h^2f_{1+n}}{13440} + \frac{3457h^2f_{\frac{3}{2}+n}}{10080} - \frac{419h^2f_{2+n}}{13440} + \frac{109h^2f_{\frac{5}{2}+n}}{6720} - \frac{19h^2f_{3+n}}{8064} \\
y_{n+\frac{5}{2}} - 5y_{\frac{1}{2}+n} &= -4y_n + \frac{1669h^2f_n}{24192} + \frac{3875h^2f_{\frac{1}{2}+n}}{4032} + \frac{5069h^2f_{1+n}}{8064} + \frac{3751h^2f_{\frac{3}{2}+n}}{6048} + \frac{1457h^2f_{2+n}}{8064} + \frac{179h^2f_{\frac{5}{2}+n}}{4032} - \frac{95h^2f_{3+n}}{24192} \\
y_{n+3} - 6y_{\frac{1}{2}+n} &= -5y_n + \frac{1375h^2f_n}{16128} + \frac{3259h^2f_{\frac{1}{2}+n}}{2688} + \frac{1489h^2f_{1+n}}{1792} + \frac{3751h^2f_{\frac{3}{2}+n}}{4032} + \frac{2059h^2f_{2+n}}{5376} + \frac{265}{896}h^2f_{\frac{5}{2}+n} + \frac{199h^2f_{3+n}}{16128} \\
hy'_n - 2y_{n+\frac{1}{2}} + 2y_n &= -\frac{28549h^2}{241920}f_n - \frac{275h^2}{1152}f_{n+\frac{1}{2}} + \frac{5717h^2}{26880}f_{n+1} - \frac{10621h^2}{60480}f_{n+\frac{3}{2}} + \frac{7703h^2}{80640}f_{n+2} - \frac{403h^2}{13440}f_{n+\frac{5}{2}} + \frac{199h^2}{48384}f_{n+3} \\
hy'_{n+\frac{1}{2}} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{275h^2}{6912}f_n + \frac{12079h^2}{40320}f_{n+\frac{1}{2}} - \frac{13823h^2}{80640}f_{n+1} + \frac{8131h^2}{60480}f_{n+\frac{3}{2}} - \frac{5771h^2}{80640}f_{n+2} + \frac{179h^2}{8064}f_{n+\frac{5}{2}} - \frac{731h^2}{241920}f_{n+3} \\
hy'_{n+1} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{2633h^2}{80640}f_n + \frac{4091h^2}{8064}f_{n+\frac{1}{2}} + \frac{17503h^2}{80640}f_{n+1} + \frac{h^2}{20160}f_{n+\frac{3}{2}} - \frac{181h^2}{16128}f_{n+2} + \frac{199h^2}{40320}f_{n+\frac{5}{2}} - \frac{h^2}{1280}f_{n+3} \\
hy'_{n+\frac{3}{2}} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{8441h^2}{241920}f_n + \frac{3907h^2}{8064}f_{n+\frac{1}{2}} + \frac{12683h^2}{26880}f_{n+1} + \frac{3751h^2}{12096}f_{n+\frac{3}{2}} - \frac{5419h^2}{80640}f_{n+2} + \frac{7h^2}{384}f_{n+\frac{5}{2}} - \frac{571h^2}{241920}f_{n+3} \\
hy'_{n+2} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{8059h^2}{241920}f_n + \frac{20071h^2}{40320}f_{n+\frac{1}{2}} + \frac{6707h^2}{16128}f_{n+1} + \frac{37507h^2}{60480}f_{n+\frac{3}{2}} + \frac{2161h^2}{11520}f_{n+2} - \frac{37h^2}{8064}f_{n+\frac{5}{2}} - \frac{29h^2}{241920}f_{n+3} \\
hy'_{n+\frac{5}{2}} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{2867h^2}{80640}f_n + \frac{3875h^2}{8064}f_{n+\frac{1}{2}} + \frac{38401h^2}{80640}f_{n+1} + \frac{1399h^2}{2880}f_{n+\frac{3}{2}} + \frac{46453h^2}{80640}f_{n+2} + \frac{8191h^2}{40320}f_{n+\frac{5}{2}} - \frac{13h^2}{1792}f_{n+3} \\
hy'_{n+3} - 2y_{n+\frac{1}{2}} + 2y_n &= \frac{1375h^2}{48384}f_n + \frac{21479h^2}{40320}f_{n+\frac{1}{2}} + \frac{1187h^2}{3840}f_{n+1} + \frac{48131h^2}{60480}f_{n+\frac{3}{2}} + \frac{15479h^2}{80640}f_{n+2} + \frac{1993h^2}{2688}f_{n+\frac{5}{2}} + \frac{36419h^2}{241920}f_{n+3}
\end{aligned} \quad (2.14)$$

111 Equation (2.14) can be written in block form as:

112

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \end{bmatrix} \\
= \begin{bmatrix} -1 & 0 \\ -2 & 0 \\ -3 & 0 \\ -4 & 0 \\ -5 & 0 \\ -2 & -h \\ -2 & 0 \\ -2 & 0 \\ -2 & 0 \\ -2 & 0 \\ -2 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} y_n \\ y_n \end{bmatrix} + \begin{bmatrix} 863h^2 \\ 48384 \\ 2803h^2 \\ 80640 \\ 2089h^2 \\ 40320 \\ 1669h^2 \\ 24192 \\ 1375h^2 \\ 16128 \\ 28549h^2 \\ 241920 \\ 275h^2 \\ 6912 \\ 2633h^2 \\ 80640 \\ 8441h^2 \\ 241920 \\ 8059h^2 \\ 241920 \\ 2867h^2 \\ 80640 \\ 1375h^2 \\ 48384 \end{bmatrix} [f_n] + h^2 \begin{bmatrix} 8999 & 769 & 1987 & 1609 & 263 & 221 \\ 40320 & 80640 & 60480 & 80640 & 40320 & 241920 \\ 1265 & 1657 & 1777 & 1049 & 11 & 137 \\ 2688 & 8960 & 20160 & 26880 & 896 & 80640 \\ 4813 & 5461 & 3457 & 419 & 109 & 19 \\ 6720 & 13440 & 10080 & 13440 & 6720 & 8064 \\ 3875 & 5069 & 3751 & 1457 & 179 & 95 \\ 4032 & 8064 & 6048 & 8064 & 4032 & 24192 \\ 3259 & 1489 & 3751 & 2059 & 265 & 199 \\ 2688 & 1792 & 4032 & 5376 & 896 & 16128 \\ 275 & 5717 & 10621 & 7703 & 403 & 199 \\ 1152 & 26880 & 60480 & 80640 & 13440 & 48384 \\ 12079 & 13823 & 8131 & 5771 & 179 & 731 \\ 40320 & 80640 & 60480 & 80640 & 8064 & 241920 \\ 4091 & 17503 & 1 & 181 & 199 & 1 \\ 8064 & 80640 & 20160 & 16128 & 40320 & 1280 \\ 3907 & 12683 & 3751 & 5419 & 7 & 571 \\ 8064 & 26880 & 12096 & 80640 & 384 & 241920 \\ 20071 & 6707 & 37507 & 2161 & 37 & 29 \\ 40320 & 16128 & 60480 & 11520 & 8064 & 241920 \\ 3875 & 38401 & 1399 & 46453 & 8191 & 13 \\ 8064 & 80640 & 2880 & 80640 & 40320 & 1792 \\ 21479 & 1187 & 48131 & 15479 & 1993 & 36419 \\ 40320 & 3840 & 60480 & 80640 & 2688 & 241920 \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix} \quad (2.15)$$

113 Now let:

$$114 \quad A^v = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h \end{bmatrix}; \quad A^x = \begin{bmatrix} -1 & 0 \\ -2 & 0 \\ -3 & 0 \\ -4 & 0 \\ -5 & 0 \\ -2 & -h \\ -2 & 0 \\ -2 & 0 \\ -2 & 0 \\ -2 & 0 \\ -2 & 0 \\ -2 & 0 \end{bmatrix}; \quad E^x = \begin{bmatrix} 863h^2 \\ 48384 \\ 2803h^2 \\ 80640 \\ 2089h^2 \\ 40320 \\ 1669h^2 \\ 24192 \\ 1375h^2 \\ 16128 \\ 28549h^2 \\ 241920 \\ 275h^2 \\ 6912 \\ 2633h^2 \\ 80640 \\ 8441h^2 \\ 241920 \\ 8059h^2 \\ 241920 \\ 2867h^2 \\ 80640 \\ 1375h^2 \\ 48384 \end{bmatrix};$$

115

$$E^v = \begin{bmatrix} 8999 & 769 & 1987 & 1609 & 263 & 221 \\ 40320 & 80640 & 60480 & 80640 & 40320 & 241920 \\ 1265 & 1657 & 1777 & 1049 & 11 & 137 \\ 2688 & 8960 & 20160 & 26880 & 896 & 80640 \\ 4813 & 5461 & 3457 & 419 & 109 & 19 \\ 6720 & 13440 & 10080 & 13440 & 6720 & 8064 \\ 3875 & 5069 & 3751 & 1457 & 179 & 95 \\ 4032 & 8064 & 6048 & 8064 & 4032 & 24192 \\ 3259 & 1489 & 3751 & 2059 & 265 & 199 \\ 2688 & 1792 & 4032 & 5376 & 896 & 16128 \\ 275 & 5717 & 10621 & 7703 & 403 & 199 \\ 1152 & 26880 & 60480 & 80640 & 13440 & 48384 \\ 12079 & 13823 & 8131 & 5771 & 179 & 731 \\ 40320 & 80640 & 60480 & 80640 & 8064 & 241920 \\ 4091 & 17503 & 1 & 181 & 199 & 1 \\ 8064 & 80640 & 20160 & 16128 & 40320 & 1280 \\ 3907 & 12683 & 3751 & 5419 & 7 & 571 \\ 8064 & 26880 & 12096 & 80640 & 384 & 241920 \\ 20071 & 6707 & 37507 & 2161 & 37 & 29 \\ 40320 & 16128 & 60480 & 11520 & 8064 & 241920 \\ 3875 & 38401 & 1399 & 46453 & 8191 & 13 \\ 8064 & 80640 & 2880 & 80640 & 40320 & 1792 \\ 21479 & 1187 & 48131 & 15479 & 1993 & 36419 \\ 40320 & 3840 & 60480 & 80640 & 2688 & 241920 \end{bmatrix}$$

116 Multiply equation (2.15) by the inverse of A^v and it yielded:

$$117 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \\ y_{n+\frac{7}{2}} \\ y_{n+4} \\ y_{n+\frac{9}{2}} \\ y_{n+5} \\ y_{n+\frac{11}{2}} \\ y_{n+6} \end{bmatrix} =$$

$$118 \begin{bmatrix} 1 & \frac{h}{2} \\ 1 & \frac{h}{2} \\ 1 & \frac{3h}{2} \\ 1 & \frac{5h}{2} \\ 1 & \frac{7h}{2} \\ 1 & \frac{9h}{2} \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ y_n \\ y_n \\ y_n \\ y_n \\ y_n \\ y_n \\ y_n \\ y_n \\ y_n \\ y_n \\ y_n \end{bmatrix} + \begin{bmatrix} 28549h^2 \\ 483840 \\ 1027h^2 \\ 7560 \\ 759h^2 \\ 3584 \\ 272h^2 \\ 945 \\ 35225h^2 \\ 96768 \\ 123h^2 \\ 280 \\ 19087h \\ 120960 \\ 1139h \\ 7560 \\ 137h \\ 896 \\ 143h \\ 945 \\ 3715h \\ 24192 \\ 41h \\ 280 \end{bmatrix} [f_n] + \begin{bmatrix} 275h^2 & 5717h^2 & 10621h^2 & 7703h^2 & 403h^2 & 199h^2 \\ 2304 & 53760 & 120960 & 161280 & 26880 & 96768 \\ 97h^2 & 2h^2 & 197h^2 & 97h^2 & 23h^2 & 19h^2 \\ 210 & 9 & 945 & 840 & 630 & 3780 \\ 1485h^2 & 2403h^2 & 45h^2 & 3267h^2 & 513h^2 & 141h^2 \\ 1792 & 17920 & 128 & 17920 & 8960 & 17920 \\ 376h^2 & 2h^2 & 656h^2 & 2h^2 & 8h^2 & 2h^2 \\ 315 & 105 & 945 & 9 & 105 & 189 \\ 8375h^2 & 3125h^2 & 25625h^2 & 625h^2 & 275h^2 & 1375h^2 \\ 5376 & 32256 & 24192 & 10752 & 2304 & 96768 \\ 27h^2 & 27h^2 & 51h^2 & 27h^2 & 27h^2 & 0 \\ 14 & 140 & 35 & 280 & 70 & 0 \\ 2713h & 15487h & 293h & 6737h & 263h & 863h \\ 5040 & 40320 & 945 & 40320 & 5040 & 120960 \\ 47h & 11h & 166h & 269h & 11h & 37h \\ 63 & 2520 & 945 & 2520 & 315 & 7560 \\ 81h & 1161h & 17h & 729h & 27h & 29h \\ 112 & 4480 & 35 & 4480 & 560 & 4480 \\ 232h & 64h & 752h & 29h & 8h & 4h \\ 315 & 315 & 945 & 315 & 315 & 945 \\ 725h & 2125h & 125h & 3875h & 235h & 275h \\ 1008 & 8064 & 189 & 8064 & 1008 & 24192 \\ 27h & 27h & 34h & 27h & 27h & 41h \\ 35 & 280 & 35 & 280 & 35 & 280 \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix} \quad (2.16)$$

119 Expand (2.16) and it yielded:

$$\left. \begin{aligned}
 y_{n+\frac{1}{2}} &= y_n + \frac{h}{2} y'_n + \frac{28549h^2}{483840d} f_n + \frac{275h^2}{2304s} f_{n+\frac{1}{2}} - \frac{5717h^2}{53760s} f_{n+1} + \frac{10621h^2}{120960s} f_{n+\frac{3}{2}} - \frac{7703h^2}{161280s} f_{n+2} + \frac{403h^2}{26880s} f_{n+\frac{5}{2}} - \frac{199h^2}{96768s} f_{n+3} \\
 y_{n+1} &= y_n + h y'_n + \frac{1027h^2}{7560d} f_n + \frac{97h^2}{210s} f_{n+\frac{1}{2}} - \frac{2h^2}{9s} f_{n+1} + \frac{197h^2}{945s} f_{n+\frac{3}{2}} - \frac{97h^2}{840s} f_{n+2} + \frac{23h^2}{630s} f_{n+\frac{5}{2}} - \frac{19h^2}{3780s} f_{n+3} \\
 y_{n+\frac{3}{2}} &= y_n + \frac{3}{2} h y'_n + \frac{759h^2}{3584d} f_n + \frac{1485h^2}{1792s} f_{n+\frac{1}{2}} - \frac{2403h^2}{17920s} f_{n+1} + \frac{45h^2}{128s} f_{n+\frac{3}{2}} - \frac{3267h^2}{17920s} f_{n+2} + \frac{513h^2}{8960s} f_{n+\frac{5}{2}} - \frac{141h^2}{17920s} f_{n+3} \\
 y_{n+2} &= y_n + 2h y'_n + \frac{272h^2}{945d} f_n + \frac{376h^2}{315s} f_{n+\frac{1}{2}} - \frac{2h^2}{105s} f_{n+1} + \frac{656h^2}{945s} f_{n+\frac{3}{2}} - \frac{2h^2}{9s} f_{n+2} + \frac{8h^2}{105s} f_{n+\frac{5}{2}} - \frac{2h^2}{189s} f_{n+3} \\
 y_{n+\frac{5}{2}} &= y_n + \frac{5}{2} h y'_n + \frac{35225h^2}{96768d} f_n + \frac{8375h^2}{5376s} f_{n+\frac{1}{2}} - \frac{3125h^2}{32256s} f_{n+1} + \frac{25625h^2}{24192s} f_{n+\frac{3}{2}} - \frac{625h^2}{10752s} f_{n+2} + \frac{275h^2}{2304s} f_{n+\frac{5}{2}} - \frac{1375h^2}{96768s} f_{n+3} \\
 y_{n+3} &= y_n + 3h y'_n + \frac{123h^2}{280d} f_n + \frac{27h^2}{14s} f_{n+\frac{1}{2}} + \frac{27h^2}{140s} f_{n+1} + \frac{51h^2}{35s} f_{n+\frac{3}{2}} + \frac{27h^2}{280s} f_{n+2} + \frac{27h^2}{70s} f_{n+\frac{5}{2}} \\
 y'_{n+\frac{1}{2}} &= y'_n + \frac{19087h}{120960d} f_n + \frac{2713h}{5040s} f_{n+\frac{1}{2}} - \frac{15487h}{40320s} f_{n+1} + \frac{293h}{945s} f_{n+\frac{3}{2}} - \frac{6737h}{40320s} f_{n+2} + \frac{263h}{5040s} f_{n+\frac{5}{2}} - \frac{863h}{120960s} f_{n+3} \\
 y'_{n+1} &= y'_n + \frac{1139h}{7560d} f_n + \frac{47h}{63s} f_{n+\frac{1}{2}} - \frac{11h}{2520s} f_{n+1} + \frac{166h}{945s} f_{n+\frac{3}{2}} - \frac{269h}{2520s} f_{n+2} + \frac{11h}{315s} f_{n+\frac{5}{2}} - \frac{37h}{7560s} f_{n+3} \\
 y'_{n+\frac{3}{2}} &= y'_n + \frac{137h}{896d} f_n + \frac{81h}{112s} f_{n+\frac{1}{2}} + \frac{1161h}{4480s} f_{n+1} + \frac{17h}{35s} f_{n+\frac{3}{2}} - \frac{729h}{4480s} f_{n+2} + \frac{27h}{560s} f_{n+\frac{5}{2}} - \frac{29h}{4480s} f_{n+3} \\
 y'_{n+2} &= y'_n + \frac{143h}{945d} f_n + \frac{232h}{315s} f_{n+\frac{1}{2}} + \frac{64h}{315s} f_{n+1} + \frac{752h}{945s} f_{n+\frac{3}{2}} + \frac{29h}{315s} f_{n+2} + \frac{8h}{315s} f_{n+\frac{5}{2}} - \frac{4h}{945s} f_{n+3} \\
 y'_{n+\frac{5}{2}} &= y'_n + \frac{3715h}{24192d} f_n + \frac{725h}{1008s} f_{n+\frac{1}{2}} + \frac{2125h}{8064s} f_{n+1} + \frac{125h}{189s} f_{n+\frac{3}{2}} + \frac{3875h}{8064s} f_{n+2} + \frac{235h}{1008s} f_{n+\frac{5}{2}} - \frac{275h}{24192s} f_{n+3} \\
 y'_{n+3} &= y'_n + \frac{41h}{280d} f_n + \frac{27h}{35s} f_{n+\frac{1}{2}} + \frac{27h}{280s} f_{n+1} + \frac{34h}{35s} f_{n+\frac{3}{2}} + \frac{27h}{280s} f_{n+2} + \frac{27h}{35s} f_{n+\frac{5}{2}} + \frac{41h}{280s} f_{n+3}
 \end{aligned} \right\} \quad (2.17)$$

120 Equation (2.17) can be written in block form as:

$$\left. \begin{aligned}
 IY_{n+\frac{i}{2}} &= AY_{n-\frac{j}{2}} + hBY'_{n-\frac{j}{2}} + h^2 \left(\frac{1}{s} CF_{n+\frac{i}{2}} + \frac{1}{d} NF_{n-\frac{j}{2}} \right) \\
 GY'_{n+\frac{i}{2}} &= TY'_{n-\frac{j}{2}} + h \left(\frac{1}{s} EF_{n+\frac{i}{2}} + \frac{1}{d} VF_{n-\frac{j}{2}} \right)
 \end{aligned} \right\} \quad (2.18)$$

121 Where 'd' and 's' are the stability parameters.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & \frac{h}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{h}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{3h}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{2h}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{5h}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{3h}{2} & 0 & 0 & 0 & 0 \end{bmatrix}, C =$$

$$= \frac{1}{s} \begin{bmatrix} \frac{275h^2}{2304} & -\frac{5717h^2}{53760} & \frac{10621h^2}{120960} & -\frac{7703h^2}{161280} & \frac{403h^2}{26880} & -\frac{199h^2}{96768} \\ \frac{97h^2}{210} & -\frac{2h^2}{9} & \frac{197h^2}{945} & -\frac{97h^2}{840} & \frac{23h^2}{630} & -\frac{19h^2}{3780} \\ \frac{1485h^2}{1792} & -\frac{2403h^2}{17920} & \frac{45h^2}{128} & -\frac{3267h^2}{17920} & \frac{513h^2}{8960} & -\frac{141h^2}{17920} \\ \frac{376h^2}{315} & -\frac{2h^2}{105} & \frac{656h^2}{945} & -\frac{2h^2}{9} & \frac{8h^2}{105} & -\frac{2h^2}{189} \\ \frac{8375h^2}{5376} & \frac{3125h^2}{32256} & \frac{25625h^2}{24192} & -\frac{625h^2}{10752} & \frac{275h^2}{2304} & -\frac{1375h^2}{96768} \\ \frac{27h^2}{14} & \frac{27h^2}{140} & \frac{51h^2}{35} & \frac{27h^2}{280} & \frac{27h^2}{70} & 0 \end{bmatrix}, N =$$

$$= \frac{1}{d} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{28549h^2}{483840} \\ 0 & 0 & 0 & 0 & 0 & \frac{1027h^2}{7560} \\ 0 & 0 & 0 & 0 & 0 & \frac{759h^2}{3584} \\ 0 & 0 & 0 & 0 & 0 & \frac{272h^2}{945} \\ 0 & 0 & 0 & 0 & 0 & \frac{35225h^2}{96768} \\ 0 & 0 & 0 & 0 & 0 & \frac{123h^2}{280} \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, E =$$

$$= \frac{1}{s} \begin{bmatrix} \frac{2713h}{5040} & -\frac{15487h}{40320} & \frac{293h}{945} & -\frac{6737h}{40320} & \frac{263h}{5040} & -\frac{863h}{120960} \\ \frac{47h}{63} & \frac{11h}{2520} & \frac{166h}{945} & -\frac{269h}{2520} & \frac{11h}{315} & -\frac{37h}{7560} \\ \frac{81h}{112} & \frac{1161h}{4480} & \frac{17h}{35} & -\frac{729h}{4480} & \frac{27h}{560} & -\frac{29h}{4480} \\ \frac{232h}{315} & \frac{64h}{2125h} & \frac{752h}{125h} & \frac{29h}{3875h} & \frac{8h}{235h} & -\frac{4h}{275h} \\ \frac{725h}{1008} & \frac{315}{8064} & \frac{125h}{189} & \frac{375h}{8064} & \frac{235h}{1008} & -\frac{275h}{24192} \\ \frac{27h}{35} & \frac{27h}{280} & \frac{34h}{35} & \frac{27h}{280} & \frac{27h}{35} & \frac{41h}{280} \end{bmatrix}, V$$

$$= \frac{1}{d} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{19087h}{120960} \\ 0 & 0 & 0 & 0 & 0 & \frac{1139h}{7560} \\ 0 & 0 & 0 & 0 & 0 & \frac{137h}{896} \\ 0 & 0 & 0 & 0 & 0 & \frac{143h}{945} \\ 0 & 0 & 0 & 0 & 0 & \frac{3715h}{24192} \\ 0 & 0 & 0 & 0 & 0 & \frac{41h}{280} \end{bmatrix}$$

122

123 where; $i \in [1, 6], j \in [5, 0]$

$$Y_{n+\frac{i}{2}} = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \end{bmatrix}, Y'_{n+\frac{i}{2}} = \begin{bmatrix} y'_{n+\frac{1}{2}} \\ y'_{n+1} \\ y'_{n+\frac{3}{2}} \\ y'_{n+2} \\ y'_{n+\frac{5}{2}} \\ y'_{n+3} \end{bmatrix}, Y_{n-\frac{j}{2}} = \begin{bmatrix} y_{n-\frac{5}{2}} \\ y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}, Y'_{n-\frac{j}{2}} = \begin{bmatrix} y'_{n-\frac{5}{2}} \\ y'_{n-2} \\ y'_{n-\frac{3}{2}} \\ y'_{n-1} \\ y'_{n-\frac{1}{2}} \\ y'_n \end{bmatrix}, F_{n+\frac{i}{2}} = \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix}, F_{n-\frac{j}{2}} = \begin{bmatrix} f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix}$$

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127 3. ANALYSIS OF THE SECOND DERIVATIVE HYBRID BLOCK METHOD(SDHBM)

128 3.1 Order and Error Constant of (SDHBM)

129 The order and error constant of the proposed method are determined from the following Truncated

130 Talyor Series Expansion of the (SDHBM)

$$\begin{aligned}
& \frac{6031y^{(9)}[x]h^9}{464486400} + \frac{18383y^{(10)}[x]h^{10}}{1061683200} + \frac{2013937y^{(11)}[x]h^{11}}{163499212800} + \frac{24225181y^{(12)}[x]h^{12}}{3923981107200} + \frac{8692795649y^{(13)}[x]h^{13}}{3570822807552000} \\
& + \frac{5719397923y^{(14)}[x]h^{14}}{7141645615104000} + O[h]^{15} \\
& \frac{233y^{(9)}[x]h^9}{7257600} + \frac{4961y^{(10)}[x]h^{10}}{116121600} + \frac{173y^{(11)}[x]h^{11}}{5702400} + \frac{116363y^{(12)}[x]h^{12}}{7664025600} + \frac{166820051y^{(13)}[x]h^{13}}{27897053184000} \\
& + \frac{438613421y^{(14)}[x]h^{14}}{223176425472000} + O[h]^{15} \\
& \frac{9y^{(9)}[x]h^9}{179200} + \frac{6129y^{(10)}[x]h^{10}}{91750400} + \frac{95729y^{(11)}[x]h^{11}}{2018508800} + \frac{383247y^{(12)}[x]h^{12}}{16148070400} + \frac{137364519y^{(13)}[x]h^{13}}{14694744064000} \\
& + \frac{90301749y^{(14)}[x]h^{14}}{29389488128000} + O[h]^{15} \\
& \frac{31y^{(9)}[x]h^9}{453600} + \frac{659y^{(10)}[x]h^{10}}{7257600} + \frac{5141y^{(11)}[x]h^{11}}{79833600} + \frac{30841y^{(12)}[x]h^{12}}{958003200} + \frac{22086637y^{(13)}[x]h^{13}}{1743565824000} + \frac{58024619y^{(14)}[x]h^{14}}{13948526592000} + O[h]^{15} \\
& \frac{1625y^{(9)}[x]h^9}{18579456} + \frac{34625y^{(10)}[x]h^{10}}{297271296} + \frac{77375y^{(11)}[x]h^{11}}{934281216} + \frac{6516325y^{(12)}[x]h^{12}}{156959244288} + \frac{468000125y^{(13)}[x]h^{13}}{28566582460416} \\
& + \frac{308271575y^{(14)}[x]h^{14}}{57133164920832} + O[h]^{15} \\
& \frac{9y^{(9)}[x]h^9}{89600} + \frac{27y^{(10)}[x]h^{10}}{204800} + \frac{727y^{(11)}[x]h^{11}}{7884800} + \frac{1431y^{(12)}[x]h^{12}}{31539200} + \frac{2013681y^{(13)}[x]h^{13}}{114802688000} + \frac{5187363y^{(14)}[x]h^{14}}{918421504000} + O[h]^{15} \\
& \frac{275y^{(9)}[x]h^8}{6193152} + \frac{55211y^{(10)}[x]h^9}{928972800} + \frac{39379y^{(11)}[x]h^{10}}{928972800} + \frac{41431y^{(12)}[x]h^{11}}{1946419200} + \frac{16490711y^{(13)}[x]h^{12}}{1961990553600} + \frac{1647295913y^{(14)}[x]h^{13}}{595137134592000} \\
& + \frac{1688619833y^{(15)}[x]h^{14}}{2142493684531200} + O[h]^{15} \\
& \frac{y^{(9)}[x]h^8}{30240} + \frac{361y^{(10)}[x]h^9}{8294400} + \frac{3557y^{(11)}[x]h^{10}}{116121600} + \frac{12961y^{(12)}[x]h^{11}}{851558400} + \frac{571y^{(13)}[x]h^{12}}{95800320} + \frac{8059109y^{(14)}[x]h^{13}}{4132896768000} \\
& + \frac{14793319y^{(15)}[x]h^{14}}{26781171056640} + O[h]^{15} \\
& \frac{9y^{(9)}[x]h^8}{229376} + \frac{603y^{(10)}[x]h^9}{11468800} + \frac{431y^{(11)}[x]h^{10}}{11468800} + \frac{9547y^{(12)}[x]h^{11}}{504627200} + \frac{60477y^{(13)}[x]h^{12}}{8074035200} + \frac{18169609y^{(14)}[x]h^{13}}{7347372032000} \\
& + \frac{6222883y^{(15)}[x]h^{14}}{8816846438400} + O[h]^{15} \\
& \frac{y^{(9)}[x]h^8}{30240} + \frac{313y^{(10)}[x]h^9}{7257600} + \frac{109y^{(11)}[x]h^{10}}{3628800} + \frac{1571y^{(12)}[x]h^{11}}{106444800} + \frac{4379y^{(13)}[x]h^{12}}{766402560} + \frac{8593579y^{(14)}[x]h^{13}}{4649508864000} \\
& + \frac{4334483y^{(15)}[x]h^{14}}{8369115955200} + O[h]^{15} \\
& \frac{275y^{(9)}[x]h^8}{6193152} + \frac{325y^{(10)}[x]h^9}{5308416} + \frac{1675y^{(11)}[x]h^{10}}{37158912} + \frac{12745y^{(12)}[x]h^{11}}{544997376} + \frac{748655y^{(13)}[x]h^{12}}{78479622144} + \frac{190605y^{(14)}[x]h^{13}}{58778976256} \\
& + \frac{81573665y^{(15)}[x]h^{14}}{85699747381248} + O[h]^{15} \\
& - \frac{9y^{(10)}[x]h^9}{716800} - \frac{27y^{(11)}[x]h^{10}}{1433600} - \frac{67y^{(12)}[x]h^{11}}{4505600} - \frac{129y^{(13)}[x]h^{12}}{15769600} - \frac{1610347y^{(14)}[x]h^{13}}{459210752000} - \frac{228453y^{(15)}[x]h^{14}}{183684300800} + O[h]^{15}
\end{aligned} \quad (3.1)$$

131

132 Equation (3.1) shows that the corresponding scheme is of order 7, 7, 7, 7, 7, 6, 6, 6, 6, 6, 7 with
 133 error constants respectively, as follows:

$$134 \left[\frac{6031}{464486400}, \frac{233}{7257600}, \frac{9}{179200}, \frac{31}{453600}, \frac{1625}{18579456}, \frac{9}{89600}, \frac{275}{6193152}, \frac{1}{30240}, \frac{9}{229376}, \frac{1}{30240}, \frac{275}{6193152}, -\frac{9}{716800} \right]^T.$$

135 **3.2 Consistency:** The block method (2.19) is consistent because its order is more than 1.

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139 3.3 Zero Stability of (SDHBM)

140 The first characteristic polynomial of (2.19) is:

$$141 \rho(r) = \det[rI - A]$$

$$\rho(r) = \det \begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & r \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{vmatrix} r & 0 & 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 & 0 & -1 \\ 0 & 0 & r & 0 & 0 & -1 \\ 0 & 0 & 0 & r & 0 & -1 \\ 0 & 0 & 0 & 0 & r & -1 \\ 0 & 0 & 0 & 0 & 0 & r-1 \end{vmatrix} = (r-1)r^5$$

142 Then, $\rho(r) = (r-1)r^5 = 0$ means that $r = 0$ or $r = 1$.

143 So, the method is zero stable.

144 To investigate the stability region, we considered the Boundary Locus Technique together with the test
145 equation given as:

$$y'' = \lambda^2 y$$

146 where $\lambda \in \mathbb{C}$

147 Now from the block method:

$$\begin{cases} IY_{n+\frac{i}{2}} = AY_{n-\frac{j}{2}} + hBY'_{n-\frac{j}{2}} + h^2 \left(\frac{1}{s} CF_{n+\frac{i}{2}} + \frac{1}{d} NF_{n-\frac{j}{2}} \right) \\ GY'_{n+\frac{i}{2}} = TY'_{n-\frac{j}{2}} + h \left(\frac{1}{s} EF_{n+\frac{i}{2}} + \frac{1}{d} VF_{n-\frac{j}{2}} \right) \end{cases}$$

148 Test equation is used to transform the above block method to:

$$\begin{cases} IY_{n+\frac{i}{2}} = AY_{n-\frac{j}{2}} + hBY'_{n-\frac{j}{2}} + \frac{1}{s} Ch^2 \lambda^2 Y_{n+\frac{i}{2}} + \frac{1}{d} Nh^2 \lambda^2 Y_{n-\frac{j}{2}} \\ GY'_{n+\frac{i}{2}} = TY'_{n-\frac{j}{2}} + \frac{1}{s} Eh \lambda^2 Y_{n+\frac{i}{2}} + \frac{1}{d} V \lambda^2 h Y_{n-\frac{j}{2}} \end{cases}$$

$$\begin{cases} IY_{n+\frac{i}{2}} = AY_{n-\frac{j}{2}} + hBY'_{n-\frac{j}{2}} + \frac{1}{s} CzY_{n+\frac{i}{2}} + \frac{1}{d} NzY_{n-\frac{j}{2}} \\ GY'_{n+\frac{i}{2}} = TY'_{n-\frac{j}{2}} + \frac{1}{s.h} EzY_{n+\frac{i}{2}} + \frac{1}{d.h} VzY_{n-\frac{j}{2}} \end{cases} \quad (3.2)$$

149 We form a coupled block method from (3.2) as:

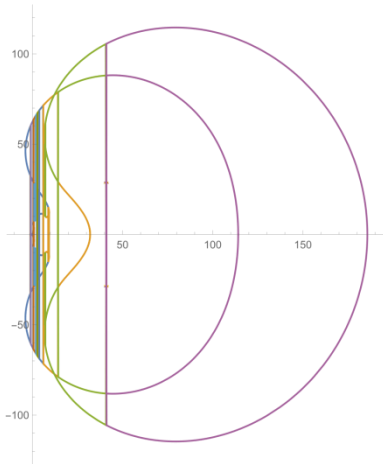
$$\begin{bmatrix} I - \frac{z}{s}C & 0 \\ -\frac{z}{s.h}E & I \end{bmatrix} \begin{bmatrix} Y_{n+\frac{i}{2}} \\ Y'_{n+\frac{i}{2}} \end{bmatrix} = \begin{bmatrix} A + zN & hB \\ \frac{z}{h.d}V & A \end{bmatrix} \begin{bmatrix} Y_{n-\frac{j}{2}} \\ Y'_{n-\frac{j}{2}} \end{bmatrix} \quad (3.3)$$

$$M(z) = \begin{bmatrix} I - \frac{z}{s}C & 0 \\ -\frac{z}{s.h}E & I \end{bmatrix}^{-1} \begin{bmatrix} A + zN & hB \\ \frac{z}{h.d}V & A \end{bmatrix} \quad (3.4)$$

Note: Matrix $A = T$ and $I = G$

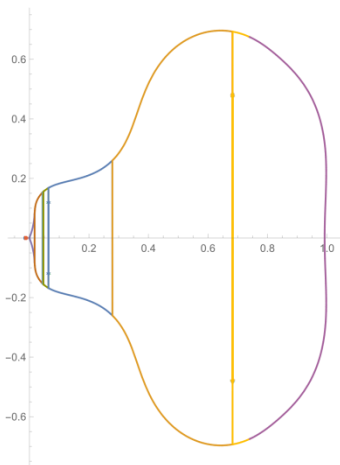
150 $M(z)$ is the amplification matrix which is used to obtain the stability plots as follows:

151



152

153 Figure1: Stability Region of SDHBM for $d = 0.9$; $s = -1.2$
154 Figure1 shows that SDHBM is $A(\alpha)$ stable.
155
156



157

158 Figure.2: Stability Region of SDHBM for $d = 1$; $s = 1$
159 Figure.2 shows that SDHBM is A-stable.
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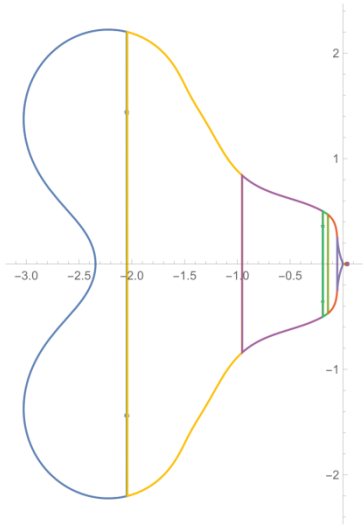


Figure.3: Stability Region of SDHBM for $d = -0.78$; $s = -0.06$

Figure 3 has shown that SDHBM is stable in a defined region $[-2.5, 0]$

4. Numerical Experiment

For experiment, we considered the following problems.

Problem 4.1: $y'' = y'$; $y(0) = 0$, $y'(0) = -1$, $h = 0.1x \in [0, 2]$

Exact solution: $y(x) = 1 - e^x$ (Kayode & Adegboro 2018).

Problem 4.2: $y'' = x(y')^2$, $y(0) = 1$, $y'(0) = 0.5$, $h = 0.01$, $x \in [0, 2]$

Exact solution: $y = 1 + \frac{1}{2} \ln\left(\frac{2+x}{2-x}\right)$ Adeyeye et al (2017)

Problem 4.3: $y'' - 2y^3 = 0$, $y(1) = 1$, $y'(1) = -1$, $h = \frac{1}{10}$, $x \in [0, 2]$

Exact solution: $y(x) = \frac{1}{x}$ Kamarun et al (2023)

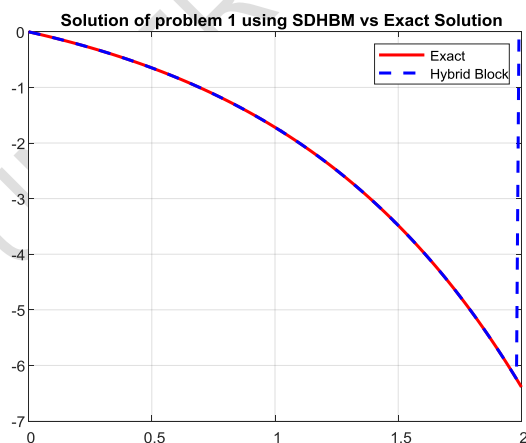


Figure 4. Solution to problem 4.1

Figure 4. is the solution to problem 4.1 which is solved directly with SDHBM and the result is compared with the exact solution and it shows competitiveness.

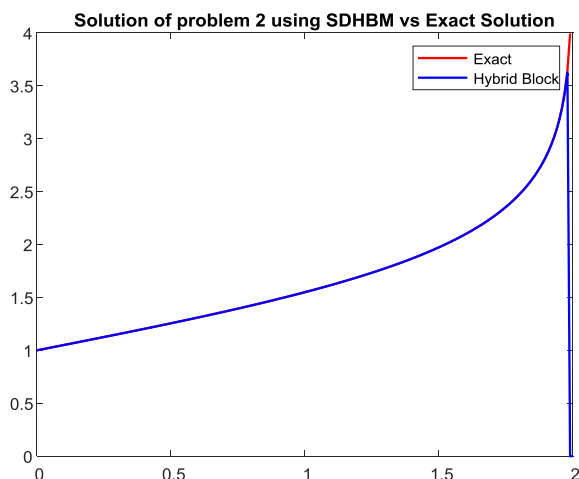


Figure 5 is a solution to problem 4.2 which is solved with SDHBM directly when the method is A-stable and the result closely approximate the exact solution. That is, zero error or negligibly small error.

Table 1: Comparison of errors between the proposed scheme and Kamarun et al (2023)

x	Error from the proposed Method	Error from Kamarun et al (2023)
1.10	5.4903e-12	2.810252e-11
1.20	1.9997e-11	4.390854e-11
1.30	4.0977e-11	2.088394e-08
1.40	6.6200e-11	4.254005e-08
1.50	9.5770e-11	6.852856e-08
1.60	1.2963e-10	9.633702e-08
1.70	1.6674e-10	1.269875e-07
1.80	2.0878e-10	1.602706e-07
1.90	2.5426e-10	1.966604e-07
2.00	3.0476e-10	2.363168e-07

Table 1 Shows the error composition of the proposed method and Kamarun et al (2023).

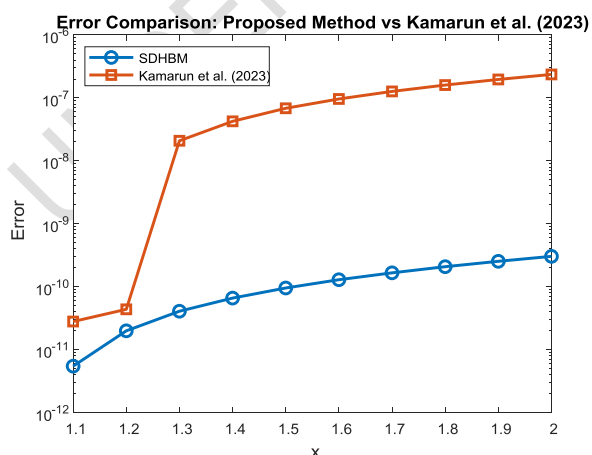


Figure 6 is the error plot of the proposed method and Kamarun et al (2023)

Discussion of results

The graphical results presented in Figures 4 and 5 for Test Problems 4 and 5, respectively, show that the numerical solutions closely overlap with the exact solutions throughout the integration interval. This indicates the high accuracy and consistency of the proposed method in approximating the exact solutions.

Furthermore, Table.1 presents the error comparison between the proposed method and the method developed by Kamarun et al.(2023) for Test Problem 4.3. The corresponding error plot, shown in Figure 6 demonstrates that the proposed method produces smaller errors, which therefore shows its efficiency, and reliability for solving IVPs in ODEs. The close agreement between both solutions further validates the accuracy and stability of the proposed method for solving the considered ordinary differential equations.

Conclusion

We have introduced the Second Derivative Hybrid Block Method (SDHBM) for the direct solution of second-order initial value problems in ordinary differential equations. The method was tested on some IVPs, and the results obtained were compared with the exact solutions as well as with the error results reported by Kamarun et al. (2023). The comparisons show that the proposed method is competitive. These results suggest that the method provides a reliable alternative solver for second-order IVPs in ODEs, with potential applications to more complex systems arising from real-world problems.

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